Choice Structures in Games

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Abstract

Following the decision-theoretic approach to game theory, we extend the analysis of [EW96] and [DT08] from hierarchies of preference relations to hierarchies of choice functions. We then construct the universal choice structure containing all these choice hierarchies, and show how the universal preference structure is embedded in it.

1 Introduction

A fundamental question in epistemic game theory is: what is the implication of supposing that each player is rational, that each player believes that the other is rational, and so on? The answer famously provided by [Ber84] and [Pea84] is that the players will choose actions that are iteratively undominated. However, the concepts of rationality, belief and iterative dominance are subject to different definitions and interpretations. Both [Ber84] and [Pea84] develop their results in the classic setting where a belief is a probability distribution over the states, rationality means choosing an action that maximizes expected utility, and an action (strongly) dominates another if it provides a higher expected payoff in all possible states.

Consider the following game, that Ann is playing as the row player and Bob as the column player.

	l	r
u	5;1	0;0
m	3;2	0;1
c	1;1	3;0
d	1;2	2;3

Ann has two dominated actions: m and d. Specifically, the mixed action $pu + (1-p)c$ strongly dominates m for $p \in (1/2, 1)$ and d for $p \in (0, 1/3)$. Therefore, no probabilistic belief on Bob's actions can justify the choice of either m or d in terms of expected utility maximization: for every probabilistic belief of Ann, either u or c will have higher expected utility than m as well as d. This is evident from Figure 1a, where the horizontal axis represents the probability of Bob playing r and the vertical axis represents Ann's expected utilities. Actions that are consistent with the players' rationality and common belief in rationality are called rationalizable. In the example, the only rationalizable action profile is (u, l), since action r is no longer rational when d is eliminated.

When, in the wake of decision-theoretic developments, classic probabilistic beliefs are generalized to possibly non-probabilistic beliefs (e.g. [GS89, Sch89]),

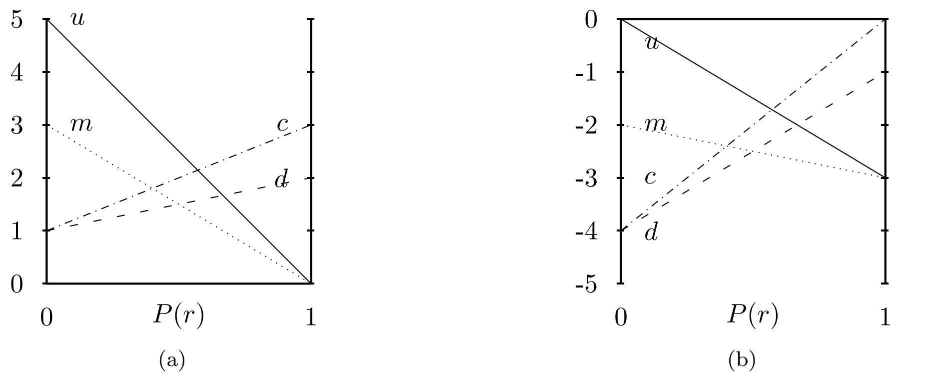


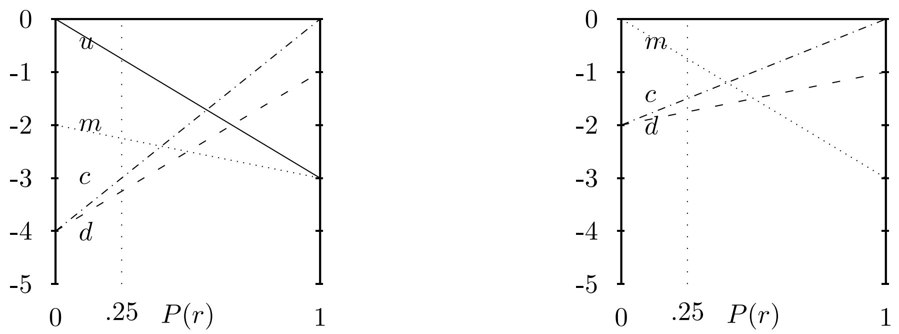
Figure 1

the game-theoretic notions of rationality, dominance and equilibrium have to be reconsidered accordingly. Examples of these advancements in game theory can be found e.g. in [Kli96, Lo96, Mar00, KU05, RS10, HP12, BCVMM15]. For the sake of explanation, let us take the case where beliefs are represented by sets of probability distributions as in the multiple-prior (MP, henceforth) model of [GS89]. In these cases, each action is associated with a set of expected utilities, and a natural notion of rational choice consists in picking an action with the highest minimal expected utility. When maxmin expected utility is substituted for expected utility maximization as the notion of rationality in the presence of non-probabilistic belief, a question naturally arises: how is dominance defined in this setting?

An answer is offered by [Eps97], who establishes a correspondence between iterated elimination of MP-dominated actions and MP-rationalizability. To exemplify, consider again the game above between Ann and Bob, and suppose that both players are expected utility maximinimizers: what are then the behavioral implications of rationality and common belief in rationality? Figure 1a shows that actions u, c and d are justifiable for Ann: u and c are still best replies to some probabilistic belief, while d is now a possible best reply to some non-probabilistic belief, e.g., the set of probability distributions that coincides with the simplex over Bob's actions. The MP-rationalizable action profiles are therefore the members of the Cartesian product $\{u, c, d\} \times \{l, r\}$.

The results about MP-dominance and MP-rationalizability in [Eps97] are based on preference structures, i.e., type spaces whose elements consist in hierarchies of interactive, reflexive and transitive preference relations over Savage-style acts. The space of all coherent preference hierarchies, i.e., the universal preference structure, is thus foundational to the results about iterated dominance and rationalizability for decision criteria that are representable by reflexive and transitive preference relations over acts, such as maxmin expected utility and all other noteworthy criteria in [Eps97]. Universal preference structures have been constructed by both [EW96] and [DT08].

Moreover, once multiple decision criteria are introduced for single-agent problems, it is natural to think of situations where different individuals adhere to different criteria in playing games. In such cases, we may have for



(a) (b)

Figure 2

instance Ann playing the game and choosing actions according to criterion A, while Bob is making his choices according to criterion B. Uncertainty about the opponent's criterion therefore enters the players' epistemic state and spreads to higher-order levels too: Ann may be uncertain about Bob's criterion and about Bob's uncertainty relative to her criterion, and so on. Preference structures provide a formal framework to express such interactive higher-order uncertainty about the players' decision criteria.

The motivation for the present work comes from the limitations imposed by employing order relations to represent the players' preferences, as already recognized by [Eps97]. Not all important decision criteria are representable by order relations and, consequently, preference structures are of no help to the study of iterated dominance and rationalizability when rationality is defined in terms of such criteria. A famous example is regret minimization, which violates the principle of independence of irrelevant alternatives and hence transitivity too. As a consequence, regret-minimizing preferences cannot be represented by order relations and have been axiomatized by means of choice functions ([Hay08] and [Sto11]).

Consider the game above again, but now suppose that both Ann and Bob are regret minimizers. Figure 1b helps picture the situation, where each action is plotted in terms of its expected negative regret. Taking advantage of the fact that minimization of (positive) regret is equivalent to maxmin negative regret, Figure 1b shows that action m is now a possible best reply (e.g, to the set of probability distributions coinciding with the simplex over Bob's actions), whereas action d is no longer a best reply to any possible belief of Ann's. Bob would consequently not play action r, and the only regret-rationalizable profile is thus (u, l).

To exemplify the reason why regret minimization violates the independence of irrelevant alternatives, we can make use of the game above again. Suppose for instance that Ann is a regret minimizer and her belief is represented by a convex compact set of probability distributions assigning action r a lower probability of 0.25 and an upper probability of 1 (see Figure 2). For this specific belief, Ann is then indifferent between actions u, m and c, when she can choose among her four original actions (Figure 2a). However, when action u is no longer available, Ann will go for action c (Figure 2b): actions m and c are no longer equivalent, in violation of the independence of irrelevant alternatives. This choice pattern cannot be encoded by a preference order, and a choice function C such that

$$C(\{u, m, c, d\}) = \{u, m, c\} \qquad C(\{m, c, d\}) = \{c\}$$

has instead to be employed. These cases never arise for maxmin expected utility, in that the relative order among possible alternatives does not change when actions are added or removed. This is essentially the reason why criteria such as maxmin expected utility can be axiomatized by preference orders ([GS89]), while context-dependent criteria like regret minimization require the use of choice functions ([Hay08] and [Sto11]).

The present paper aims to build the interactive epistemic structures required to take into account cases of decision criteria like regret minimization, where preference orders are insufficient to represent the players' rationality. Such motivation justifies the shift from hierarchies of interactive preference orders to hierarchies of interactive choice functions, that is, the shift from preference structures to choice structures. In doing so, we define here the notion of choice structure and build the universal choice structure, i.e., the structure containing all coherent hierarchies of choice functions.

Apart from the philosophical and conceptual motivation of this work, choice structures turn out to be the most general models for interactive epistemology and epistemic game theory introduced so far. Since preference orders are special cases of choice functions, the universal choice structure embeds the universal preference structure, as we show later, which in turn embeds the universal type structure. On a more formal side, moreover, the construction of the universal choice structure makes use of a toolkit from the theory of coalgebras and category theory, which may also shed new light on the construction of both the universal type structure and the universal preference structure.

The paper is structured as follows. Section 2 introduces the setting and the preliminary notions that are used in Section 3 to construct hierarchies of choice functions and the universal choice structure. Section 4 shows how the universal preference structure is embedded into the universal choice structure, and Section 5 concludes.

2 Preliminaries

In this section we introduce the mathematical notions that will be employed in game-theoretic contexts in the following sections.

2.1 Uncertainty spaces

We are working with *uncertainty spaces*, or simply *spaces*, $(X, \mathcal{B})$, where X is any set, whose elements are called *states*, and $\mathcal{B} \subseteq 2^X$ is an algebra of subsets of X, whose elements are called *measurable sets* or *events*.[1] We often write just X for an uncertainty space $(X, \mathcal{B})$. We implicitly take every finite set Z to be

[1] That $\mathcal{B}$ is an algebra means that it is closed under finite intersections and complements. Thus, an uncertainty space $(X, \mathcal{B})$ is almost a measurable space, but without requiring closure under countable intersections.

an uncertainty space $(Z, \mathcal{B})$ in which every subset is measurable, meaning that $\mathcal{B} = 2^Z$ is the discrete algebra containing all subsets of Z.

A morphism φ from an uncertainty space X to an uncertainty space Y is any function $\varphi : X \to Y$ that is *measurable*, meaning that $\varphi^{-1}[E] = \{x \in X \mid \varphi(x) \in E\}$ is measurable in X whenever E is measurable in Y. For every uncertainty space $X = (X, \mathcal{B})$ we use id_X to denote the measurable identity function on X that is $\mathsf{id}_X : X \to X, x \mapsto x$.

Given two uncertainty spaces X and Y we can define their product to be the uncertainty space $X \times Y$ whose states are all pairs (x, y) where the first component is a state in X and the second component is a state in Y. The measurable sets of states are generated by taking finite unions and complements of cylinders, that is, sets of the form $U \times Y$ and $X \times V$ for measurable U in X and V measurable in Y. It is clear that with this algebra the projections $\pi_1 : X \times Y \to X, (x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y, (x, y) \mapsto y$ are measurable functions. Moreover given two measurable functions $\varphi : X \to X'$ and $\psi : Y \to Y'$ we use $\varphi \times \psi : X \times Y \to X' \times Y'$ to denote the measurable function, which is defined such that $(\varphi \times \psi)(x, y) = (\varphi(x), \psi(y))$ for all $(x, y) \in X \times Y$. It is not hard to check that this $\varphi \times \psi$ is indeed measurable.

2.2 Acts

Fix a finite non-empty set Z of *outcomes*. An *act* for an uncertainty space X is a measurable function $f : X \to Z$. Using that the measurable sets in X are closed under finite unions one easily checks that a function $f : X \to Z$ for a finite Z, with the discrete algebra, is measurable precisely if $f^{-1}[\{z\}]$ is measurable for all $z \in Z$. We write FX for the set of all acts for some uncertainty space X. For every measurable function $\varphi : X \to Y$ we obtain a function $F\varphi : FY \to FX, f \mapsto f \circ \varphi$.

2.3 Choice functions

A *choice function* over a set X is a function C that maps every finite subset $F \subseteq X$ to one of its subsets $C(F) \subseteq F$. For any set X we write $\mathcal{C}X$ for the set of all choice functions over X. Even if X is just a set, without a notion of measurable subset, we take $\mathcal{C}X$ to be the uncertainty space in which a set is measurable if it can be generated by taking finite intersections and complements of sets of the form

$$B_L^K = \{C \in \mathcal{C}X \mid C(K) \subseteq L\},$$

for some finite $K, L \subseteq X$ with $L \subseteq K$.

Given any function $f : X \to Y$ we define the function $\mathcal{C}f : \mathcal{C}Y \to \mathcal{C}X$ by setting $\mathcal{C}f(C) = C^f$, where C^f is the choice function mapping a finite $K \subseteq X$ to

$$C^f(K) = f^{-1}[C(f[K])] \cap K.$$

We use $f[K] \subseteq Y$ to denote the direct image $f[K] = \{f(x) \in Y \mid x \in K\}$ of K. One can show that the function $\mathcal{C}f : \mathcal{C}Y \to \mathcal{C}X$ is measurable. To see this one first checks that for all measurable sets of the form B_L^K with $K \subseteq L$

$$\begin{aligned}(\mathcal{C}f)^{-1}[B_L^K] &= \{C \in \mathcal{C}Y \mid f^{-1}[C(f[K])] \cap K \subseteq L\} \\ &= \{C \in \mathcal{C}Y \mid C(f[K]) \subseteq \hat{L}\} = B_{\hat{L}}^{f[K]},\end{aligned}$$

where $\hat{L} = \{y \in f[K] \mid f^{-1}[\{y\}] \cap K \subseteq L\}$. This then extends to arbitrary measurable sets because intersections and complements are preserved under taking inverse images.

2.4 Choice functions over acts

We then consider choice functions over acts for some uncertainty space X. For every uncertainty space X we define the uncertainty space $\Gamma X = \mathcal{C}FX$ to be the uncertainty space of all choice functions over acts for X. Moreover, for every measurable function $\varphi : X \to Y$ we can define the measurable function $\Gamma\varphi = \mathcal{C}F\varphi : \Gamma X \to \Gamma Y$. Unfolding the definitions this means that for every $C \in \Gamma X$ we have the choice function $\Gamma\varphi(C) = C^\varphi$ is such that

$$C^\varphi(K) = \{f \in K \mid f \circ \varphi \in C(\{g \circ \varphi \mid g \in K\})\}$$

for all finite sets of acts $K \subseteq FY$.

3 Choice structures

In this section, the notions introduced above are applied to the game-theoretic context that we are interested in. To keep things simple, we focus here on interactive situations with only two players, Ann and Bob. It is straightforward to adapt our setting to more than two players, but this would introduce additional notational complications.

3.1 Choice Structures

As we are working with two-player games, the basic uncertainty of Ann is just over the fixed finite set S_b of Bob's strategies and, similarly, the basic uncertainty of Bob is over the fixed finite set of Ann's strategies S_a. We thus obtain the following definition of a choice structure:

Definition 1. A *choice structure* is a tuple $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ consisting of:

- uncertainty spaces T_a and T_b of *types* for Ann and Bob,
- measurable functions $\theta_a : T_a \to \Gamma(S_b \times T_b)$ and $\theta_b : T_b \to \Gamma(S_a \times T_a)$.

A *morphism* $\alpha : \mathcal{X} \to \mathcal{X}'$ from a choice structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ to a choice structure $\mathcal{X}' = (T'_a, T'_b, \theta'_a, \theta'_b)$ consists of two measurable functions $\alpha_a : T_a \to T'_a$ and $\alpha_b : T_b \to T'_b$ such that

$$\theta'_a \circ \alpha_a = \Gamma(\mathsf{id}_{S_b} \times \alpha_b) \circ \theta_a \text{ and } \theta'_b \circ \alpha_b = \Gamma(\mathsf{id}_{S_a} \times \alpha_a) \circ \theta_b.$$

A *state* in $\mathcal{X}$ is a tuple $(s_a, s_b, t_a, t_b) \in S_a \times S_b \times T_a \times T_b$. We also say that a *state of Ann* is a pair $(s_a, t_a) \in S_a \times T_a$ and a *state of Bob* is a pair $(s_b, t_b) \in S_b \times T_b$. A state of a player therefore consists of a strategy that she is actually playing and a type that represents her "mental state". An *act of Ann* is then a map $S_b \times T_b \to Z$ from states of Bob to outcomes, and similarly for Bob. We hence model the type of a player by her choices between acts whose outcome depends on the state of the other player, but not on her own state. We remark here that this is different from the setting in [DT08], and it is related to the introspection properties of the two frameworks. In our construction, players are introspective in the sense of not having uncertainty about their own state. In the structure of [DT08], on the contrary, players may be uncertain about their own strategies and types at a state.[2] The definitions and results of this paper could be easily adapted to the case where the players are uncertain about their own type and strategy.

Example 1. We provide an example of a choice structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ for the game from the introduction. Consider again the game from the introduction, and a choice structure where Ann has a single type $T_a = \{t_a\}$, while Bob has two possible types $T_b = \{t_{Mm}, t_{EU}\}$. Here, we interpret Ann's type t_a as a regret minimizer with belief represented as in Figure 2, i.e., a convex compact set of probability distributions assigning action r lower probability of 0.25 and upper probability of 1, while Bob's type t_{Mm} is a maximinimizer with full uncertainty, i.e., not excluding any probability distribution over Ann's strategies, and Bob's type t_{EU} is an expected utility maximizer assigning probability 1/2 to Ann playing u and 1/2 to Ann playing d. The states of the choice structure are given by the Cartesian set $S_a \times S_b \times T_a \times T_b$, i.e.,

$$\{u, m, c, d\} \times \{l, r\} \times \{t_a\} \times \{t_{Mm}, t_{EU}\}.$$

Ann's states are then members of the set $\{u, m, c, d\} \times \{t_a\}$ and Bob's states are members of the set $\{l, r\} \times \{t_{Mm}, t_{EU}\}$. The set of outcomes Z is naturally given by the outcomes of the game,

$$Z = \{(5, 1), (3, 2), (1, 1), (1, 2), (0, 0), (0, 1), (3, 0), (2, 3)\}.$$

The map θ_a then associates each of Ann's types with a choice function over acts defined on Bob's states. In the running example, such acts are

$$\begin{array}{lll} f_u(l, t) = (5, 1) & f_u(r, t) = (0, 0) & \text{for both } t \in T_b \\ f_m(l, t) = (3, 2) & f_m(r, t) = (0, 1) & \text{for both } t \in T_b \\ f_c(l, t) = (1, 1) & f_c(r, t) = (3, 0) & \text{for both } t \in T_b \\ f_d(l, t) = (1, 2) & f_d(r, t) = (2, 3) & \text{for both } t \in T_b \end{array}$$

Similarly, Bob's acts are the following:

$$\begin{array}{ll} f_l(u, t_a) = (5, 1) & f_r(u, t_a) = (0, 0) \\ f_l(m, t_a) = (3, 2) & f_r(m, t_a) = (0, 1) \\ f_l(c, t_a) = (1, 1) & f_r(c, t_a) = (3, 0) \\ f_l(d, t_a) = (1, 2) & f_r(d, t_a) = (2, 3) \end{array}$$

When Ann's type t_a is a regret minimizer as described above, the choice function $C_a = \theta_a(t_a)$ associated with t_a maps subsets of the set of acts defined above as follows:

$$\begin{array}{cc} C_a(\{f_u, f_m, f_c, f_d\}) = \{f_u, f_m, f_c\} & C_a(\{f_m, f_d\}) = \{f_d\} \\ C_a(\{f_u, f_m, f_c\}) = \{f_u, f_m\} & C_a(\{f_c, f_d\}) = \{f_c\} \\ C_a(\{f_u, f_m, f_d\}) = \{f_u, f_m\} & C_a(\{f_u, f_m\}) = \{f_u\} \\ C_a(\{f_u, f_c, f_d\}) = \{f_u\} & C_a(\{f_u, f_d\}) = \{f_u\} \\ C_a(\{f_m, f_c, f_d\}) = \{f_c\} & C_a(\{f_u, f_c\}) = \{f_u, f_c\} \\ C_a(\{f_m, f_c\}) = \{f_c\} & \end{array}$$

[2] See Sections 3.1 and 3.2 in [DT08].

where we dispense with specifying the choice function in trivial cases such as singletons. The definition of a choice structure would also require C_a to be defined on all other subsets of $F(S_b \times T_b)$. For brevity we give the definition of C_a only on subsets of $\{f_u, f_m, f_c, f_d\}$, which are the acts corresponding to strategies in the game. As for Bob, we have that type t_{Mm} is associated with the choice function

$$C_{Mm}(\{f_l, f_r\}) = \{f_l\}$$

and type t_{EU} with the choice function

$$C_{EU}(\{f_l, f_r\}) = \{f_l, f_r\}.$$

Again, we omit the definition of these choice functions on sets of acts which do not correspond to strategies in the game.

3.2 Choice hierarchies and the universal choice structure

We now introduce hierarchies of choice functions that represent the higher-order attitudes of the players. To this aim we define uncertainty spaces representing the players' n-th order attitudes by a mutual induction on a and b. In the base case we set $\Omega_{a,1} = \Gamma S_b$ and $\Omega_{b,1} = \Gamma S_a$, and for the inductive step $\Omega_{a,n+1} = \Gamma(S_b \times \Omega_{b,n})$ and $\Omega_{b,n+1} = \Gamma(S_a \times \Omega_{a,n})$. The intuition is that the players' first order attitudes are represented by their choices between acts whose outcome depends just on the actual strategy played by the opponent. Players' $n+1$-th order attitudes are represented by their choices between acts, whose outcome depends on the actual strategy played by the opponent and the n-th order attitudes of the opponent.

Note that the players' $(n+1)$-th order attitudes determine their n-th order attitudes. At the first level this means that the agent's choices in $\Omega_{a,1}$ between acts that depend on the opponent's strategy are the same as her choices between acts in $\Omega_{a,2}$, when they are taken as additionally depending trivially on the opponent's first-order attitudes. This can be made precise with a measurable *coherence morphism* $\xi_{a,1} = \Gamma\pi_1 : \Omega_{a,2} \to \Omega_{a,1}$, where $\pi_1 : S_b \times \Omega_{b,1} \to S_b$ is the projection onto the first component. When $o_2 \in \Omega_{a,2}$ represents Ann's second-order attitudes then $\xi_{a,1}(o_2) \in \Omega_{a,1}$ represents her first-order attitudes.

Analogously, we define a coherence morphism for Bob, by setting $\xi_{b,1} = \Gamma\pi_1 : \Omega_{b,2} \to \Omega_{b,1}$, where $\pi_1 : S_b \times \Omega_{b,1} \to S_b$. From now on we will not bother with writing every equation explicitly for Ann on Bob. We just write the version for Ann and then write "and similarly for Bob", thereby meaning that the equation also holds with a and b interchanged.

By induction we can extend the idea of coherence to the higher levels. Choices between acts depending on the opponent's strategy and n-th order attitudes of the opponent are determined by choices between the same acts taken as depending on the opponent's strategies and their $(n+1)$-th order attitudes. Hence, we define by mutual induction

$$\xi_{a,n+1} = \Gamma(\mathsf{id}_{S_b} \times \xi_{b,n}) : \Gamma(S_b \times \Omega_{b,n+1}) \to \Gamma(S_b \times \Omega_{b,n}),$$

and similarly for Bob $\xi_{b,n+1} = \Gamma(\mathsf{id}_{S_a} \times \xi_{a,n})$.

In the limit one can then consider countable sequences $o = (o_1, o_2, \dots, o_n, \dots)$ with $o_n \in \Omega_{a,n}$ for all $n \in \omega$, which are coherent in the sense that $\xi_{a,n}(o_{n+1}) =$

o_n for all n. One such sequence completely describes a coherent state of Ann's higher-order attitudes at all levels. Let Ω_a be the infinite set of all such coherent sequences. There are projections $\zeta_{a,n} : \Omega_a \to \Omega_{a,n}$ for every level $n \in \omega$. The set Ω_a becomes a measurable space when endowed with the algebra generated from all the subsets of the form $(\zeta_{a,n})^{-1}[O_n]$ for $n \in \omega$ and measurable $O_n \subseteq \Omega_{a,n}$. All these notions can also be defined analogously for Bob.

In the appendix we prove the central result about this construction, which is that there exist the following isomorphisms:

Theorem 1. $\Omega_a \simeq \Gamma(S_b \times \Omega_b)$ and $\Omega_b \simeq \Gamma(S_a \times \Omega_a)$.

The measurable functions $\mu_a : \Omega_a \to \Gamma(S_b \times \Omega_b)$ and $\mu_b : \Omega_b \to \Gamma(S_a \times \Omega_a)$ that witness the isomorphisms from Theorem 1 allow us to define the universal choice structure:

Definition 2. The *universal choice structure* $\mathcal{U} = (\Omega_a, \Omega_b, \mu_a, \mu_b)$ consists of the uncertainty space Ω_a and Ω_b of all coherent sequences of choice attitudes, together with the measurable functions $\mu_a : \Omega_a \to \Gamma(S_b \times \Omega_b)$ and $\mu_b : \Omega_b \to \Gamma(S_a \times \Omega_a)$ from Theorem 1.

There is a technical difference between our presentation and the approach that is usually taken in the literature, such as for instance [MZ85, BD93, EW96, DT08]. It is common to define the $(n+1)$-th level $\Omega_{a,n+1}$ as consisting of all pairs $(x, y) \in \Omega_{a,n} \times \Gamma\Omega_{b,n}$ that are coherent in the sense that the attitudes represented by x are consistent with the attitudes represented by y, in a sense similar to our coherence morphisms $\xi_{a,n}$. In the limit one then considers sequences $(o_1, o_2, \dots)$ such that $o_i \in \Gamma\Omega_{b,i}$ for all $i \in \omega$. Our approach is equivalent to this approach, once coherence of the whole infinite sequences is taken into account.

3.3 Universality of the universal choice structure

Every type $t \in T_a$ in any choice structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ generates a coherent sequence of attitudes in Ω_a. To see it, let us define first $\upsilon_{a,1} = \Gamma\pi_1 \circ \theta_a : T_a \to \Omega_{a,1}, t \mapsto \Gamma\pi_1(\theta_a(t))$, where $\pi_1 : S_b \times T_b \to S_b$ is the projection. Similarly we define $\upsilon_{b,1} : T_b \to \Omega_{b,1}$. We can then continue by mutual induction and set

$$\upsilon_{a,n+1} = \Gamma(\mathsf{id}_{S_b} \times \upsilon_{b,n}) \circ \theta_a : T_a \to \Omega_{a,n+1},$$
$$t \mapsto \Gamma(id_{S_b} \times \upsilon_{b,n})(\theta_a(t)).$$

Similarly we define $\upsilon_{b,n+1} : T_b \to \Omega_{b,n+1}$.

One can easily verify that $\upsilon_{a,n} = \xi_{a,n} \circ \upsilon_{a,n+1}$ for all n. Hence, for each $t \in T_a$ the infinite sequence $(\upsilon_{a,1}(t), \upsilon_{a,2}(t), \dots)$ is coherent and we obtain a measurable map $\upsilon_a : T_a \to \Omega_a$. Similarly we also obtain a measurable map $\upsilon_b : T_b \to \Omega_b$. In the appendix we show that υ_a and υ_b together define a unique morphism υ into the universal choice structure:

Theorem 2. For every choice structure $\mathcal{X}$ there is a unique morphism of choice structures $\upsilon : \mathcal{X} \to \mathcal{U}$ from $\mathcal{X}$ to the universal choice structure $\mathcal{U}$.

4 Embedding preference structures

In this section we discuss how our hierarchies of choice functions relate to the hierarchies of preference relations introduced in [DT08].

4.1 Di Tillio's preference structures

We start by reviewing the approach by [DT08] in our notation. The fundamental notion of [DT08] is that of a preference relation over a set X. In the following a *preference relation* $\preccurlyeq$ over X is a poset, that is a reflexive, transitive and anti-symmetric relation, over the set X. We require preference relations to be anti-symmetric. This is different from [DT08], who requires preference relations to be just preorders, that means reflexive and transitive, but not necessarily anti-symmetric relations. We justify this apparent loss of generality in Remark 1 below. In most of our arguments anti-symmetry does not play a role and hence they also work for preorders. Our reason to require anti-symmetry is that in the case of preorders the embedding from preference relations into choice functions need not be injective.

Write $\mathcal{P}X$ for the set of all preference relations over the set X. The set $\mathcal{P}X$ can be turned into an uncertainty space by generating the algebra of measurable events from sets of the form $B_{x_1 \preccurlyeq x_2} = \{\preccurlyeq\, \in \mathcal{P}X \mid x_1 \preccurlyeq x_2\}$ for some $x_1, x_2 \in X$.

Every function $f : X \to Y$ gives rise to the measurable function $\mathcal{P}f : \mathcal{P}Y \to \mathcal{P}X$, where a preference relation $\preccurlyeq$ over Y maps to the preference relation $\mathcal{P}f(\preccurlyeq) = \preccurlyeq^f$ over Y that is defined by

$$x_1 \preccurlyeq^f x_2 \qquad \text{iff} \qquad f(x_1) \preccurlyeq f(x_2). \tag{1}$$

We can then redo all the constructions from Section 3 using $\mathcal{P}$ instead of $\mathcal{C}$. Let us sketch how this works.

One considers preference relations over acts by considering for every uncertainty space X the space $\Pi X = \mathcal{P}FX$. For every measurable function $\varphi : X \to Y$ we obtain the measurable function $\Pi\varphi = \mathcal{P}F\varphi : \Pi X \to \Pi Y$ which maps a preference relation $\preccurlyeq$ over FX to the preference relation $\preccurlyeq^\varphi$ over FY that is defined such that $f \preccurlyeq^\varphi g$ iff $f \circ \varphi \preccurlyeq g \circ \varphi$.

Lastly, define a *preference structure* to be a tuple $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ where $\theta_a : T_a \to \Pi(S_b \times T_b)$ and $\theta_b : T_b \to \Pi(S_a \times T_a)$ are measurable functions. A *morphism* $\alpha : \mathcal{X} \to \mathcal{X}'$ from a preference structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ to a preference structure $\mathcal{X}' = (T'_a, T'_b, \theta'_a, \theta'_b)$ consists of two measurable functions $\alpha_a : T_a \to T'_a$ and $\alpha_b : T_b \to T'_b$ such that

$$\theta'_a \circ \alpha_a = \Pi(\mathsf{id}_{S_b} \times \alpha_b) \circ \theta_a \text{ and } \theta'_b \circ \alpha_b = \Pi(\mathsf{id}_{S_a} \times \alpha_a) \circ \theta_b.$$

Note that this definition of a preference structure differs from the one given in [DT08] with respect to the assumption of introspection mentioned in Section 3.1.

The universal preference structure $\mathcal{U}' = (\Omega'_a, \Omega'_b, \mu'_a, \mu'_b)$ can be defined from the limits Ω'_a and Ω'_b of analogous sequences $(\Omega'_{a,n}, \xi'_{a,n})_{n\in\omega}$ and $(\Omega'_{b,n}, \xi'_{b,n})_{n\in\omega}$ as those that are used to approximate the universal choice structure in the previous section. Hence, set $\Omega'_{a,1} = \Pi S_b$, $\Omega'_{b,1} = \Pi S_a$ and then inductively $\Omega'_{a,n+1} = \Pi(S_b \times \Omega'_{b,n})$ and $\Omega'_{b,n+1} = \Pi(S_a \times \Omega'_{a,n})$. The coherence morphism are such that $\xi'_{a,1} = \Pi\pi_1$, $\xi'_{b,1} = \Pi\pi_2$ and inductively $\xi'_{a,n+1} = \Pi(\mathsf{id}_{S_b} \times \xi'_{b,n})$ and $\xi'_{b,n+1} = \Pi(\mathsf{id}_{S_a} \times \xi'_{a,n})$. The existence of suitable μ'_a and μ'_b and universality properties of $\mathcal{U}'$ then follow from a construction that is analogous to the one given in Appendix C, using Π in place of Γ. The properties of Π that are required for this construction to succeed are stated in Section 3 of [DT08].

4.2 Maximization

We use maximization to map preference orders to choice functions. Given a finite set of alternatives, a player with a given preference relation chooses the most preferred alternatives of the set. Formally, this means that given a preference relation $\preccurlyeq \in \mathcal{P}X$ over a set X we map it to the choice function $m_X(\preccurlyeq) \in \mathcal{C}X$ that assigns to a finite set $K \subseteq X$ the set $m_X(\preccurlyeq)(K)$ of its maximal elements, which is defined as follows:

$$m_X(\preccurlyeq)(K) = \{m \in K \mid \text{there is no } k \in K \text{ with } m \prec k\},$$

where $x \prec y$ is defined to hold iff $x \preccurlyeq y$ and not $y \preccurlyeq x$. This definition yields a measurable function $m_X : \mathcal{P}X \to \mathcal{C}X$ for every set X. To prove that it is measurable it suffices to show that for every basic event $B^K_L = \{C \in \mathcal{C}X \mid C(K) \subseteq L\}$ of $\mathcal{C}X$ its preimage $m_X^{-1}[B^K_L]$ is measurable in $\mathcal{P}X$. To see that this is the case first observe that the set of maximal elements of a set K in a preference relation $\preccurlyeq$ is a subset of L iff for every element $k \in K \setminus L$ there is some $l \in L$ such that $k \preccurlyeq l$ and not $l \preccurlyeq k$. Thus we can write

$$m_X^{-1}[B^K_L] = \bigcap_{k \in K \setminus L} \bigcup_{l \in L} (B_{k \preccurlyeq l} \setminus B_{l \preccurlyeq k}) \,.$$

Since K, L and hence also $K \setminus L$ are finite the right side of the above equality is a finite intersection of finite unions of intersections of basic events with complements of basic events.

Remark 1. With our definition of the choice function $m_X(\preccurlyeq)$ from the preference relation $\preccurlyeq$ we do not lose any generality by restricting to posets instead of preorders. For every preorder $\preccurlyeq$ we can define the poset $\preccurlyeq'$ by setting $x \preccurlyeq' x'$ iff $x = x'$ or, $x \preccurlyeq x'$ and not $x' \preccurlyeq x$. One can show that $\preccurlyeq'$ is anti-symmetric and that the maximal elements of any finite set $K \subseteq X$ in $\preccurlyeq'$ are the same as the maximal elements of K in $\preccurlyeq$, meaning that $m_X(\preccurlyeq')(K) = m_X(\preccurlyeq)(K)$. It follows from this that any choice function that arises from maximization in some preorder also arises from maximization in some poset. Hence, the restriction to posets does not lose any generality in the class of choice behaviors that we can account for.

There are examples in which the preorders $\preccurlyeq$ and $\preccurlyeq'$ are distinct. For instance, we might take $X = \{x_1, x_2\}$ to be a two element set and $\preccurlyeq$ the total relation, in which the two elements of X are equally preferred, meaning that $x_1 \preccurlyeq x_2$ and $x_2 \preccurlyeq x_1$. Applying the definition of $\preccurlyeq'$ from above shows that $\preccurlyeq'$ is then the poset in which x_1 and x_2 are incomparable, meaning that neither $x_1 \preccurlyeq' x_2$ nor $x_2 \preccurlyeq' x_1$. We have then that $m_X(\preccurlyeq)(K) = K = m_X(\preccurlyeq')(K)$ for every $K \subseteq X$. Hence there exist two preorders that account for the same choice behavior. We show in the following proposition that this redundancy disappears if one restricts to posets.

The next proposition shows that any two distinct preference relations give rise via maximization to distinct choice functions. We show this proposition in the appendix. It relies on our assumption that preference relations are anti-symmetric.

Proposition 1. The measurable function $m_X : \mathcal{P}X \to \mathcal{C}X$ is injective for all sets X.

We can extend maximization to preference relations and choice functions over acts by defining for every space X the measurable function $\lambda_X = m_{FX} : \Pi X \to \Gamma X$. A crucial property of λ_X is stated in the following proposition, which we prove in the appendix:

Proposition 2. The mapping λ_X is natural in X. This means that for every measurable function $\varphi : X \to Y$ we have that $\lambda_Y \circ \Pi\varphi = \Gamma\varphi \circ \lambda_X$.

4.3 Embedding preference structures into choice structures

We can use the λ to turn preference structures into choice structures. For every preference structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$ define the choice structure $\lambda(\mathcal{X}) = (T_a, T_b, \lambda_{S_b \times T_b} \circ \theta_a, \lambda_{S_a \times T_a} \circ \theta_b)$. It is an easy consequence of Proposition 2 that this embedding preserves morphisms in the sense that whenever $\chi : \mathcal{X} \to \mathcal{X}'$ is a morphism of preference structures then the same pair of measurable functions is also a morphism of choice structures $\chi : \lambda(\mathcal{X}) \to \lambda(\mathcal{X}')$.

It is also possible to characterize the class of choice structures $\lambda(\mathcal{X}) = (T_a, T_b, \lambda_{S_b \times T_b} \circ \theta_a, \lambda_{S_a \times T_a} \circ \theta_b)$ arising from some preference structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$. To this end, one can use any of the axiomatizations of the class $\mathbb{P}$ of choice functions that arise from the maximization in a poset. Such an axiomatization is given for instance in Theorem 2.9 of [ABM07]. We then have that a choice structure $\mathcal{X}' = (T'_a, T'_b, \theta'_a, \theta'_b)$ is equal to $\lambda(\mathcal{X})$ for some preference structure $\mathcal{X}$ iff for all types $t_a \in T'_a$ and $t_b \in T'_b$ the choice functions $\theta_a(t_a) \in \Gamma(S_b \times T'_b)$ and $\theta_b(t_b) \in \Gamma(S_a \times T'_a)$ are in $\mathbb{P}$.

Because λ_X is injective for every space X, it follows that whenever $\lambda(\mathcal{X}) = \lambda(\mathcal{X}')$ for preference structures $\mathcal{X}$ and $\mathcal{X}'$ then $\mathcal{X} = \mathcal{X}'$. Therefore, any difference between preference structures is preserved in choice structures. One can also define an injective embedding of the universal preference structure into the universal choice structure. Consider the unique morphism $\upsilon : \lambda(\mathcal{U}') \to \mathcal{U}$ of choice structures from $\lambda(\mathcal{U}')$ to $\mathcal{U}$ that exists by Theorem 2. This morphism consists of two measurable functions $\upsilon_a : \Omega'_a \to \Omega_a$ and $\upsilon_b : \Omega'_b \to \Omega_b$ for which we argue in the appendix that it is injective if S_a and S_b are finite. Hence, we obtain the following theorem:

Theorem 3. Assume that S_a and S_b are finite. Then the uncertainty spaces Ω'_a and Ω'_b of all preference hierarchies are isomorphic to two subspaces of the spaces of all choice hierarchies Ω_a and Ω_b respectively.

5 Conclusion and further work

Differently from [EW96] and [DT08], in this paper we take choice functions rather than preference relations as primitives for describing the decisions of the players. Our main result is the construction of the universal structure of all hierarchies of choice functions, and we then show how hierarchies of preference relations are embedded in this universal choice structure.

Similar to [DT08], here we mostly work with finite sets, in particular it is crucial that the outcome set Z is finite. We leave for future investigation to extend the analysis to the infinite case by considering for instance compact Hausdorff spaces of outcomes as in [EW96].

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A Preliminaries

The appendix contains technical proofs about the constructions of the universal choice structure from Section 3 and the embedding of the universal preference structure into the universal choice structure from Section 4. We view these constructions as instances of more general constructions from the theory of coalgebras [Jac16]. Thereby, we follow the approach laid out in [MV04], who show how classical probabilistic type spaces can be seen as coalgebras.

We start off by explaining the notion of a coalgebra and how choice structures are instances of this concept. We first need to introduce some basic notions from category theory [Lan98, Awo06].

A.1 Categories

A *category* $\mathcal{C}$ is a collection of objects, possibly a proper class, together with a collection of morphisms, written as $f : X \to Y$, between any two objects X and Y such that for every object X in $\mathcal{C}$ there is an *identity* morphism $\mathsf{id}_X : X \to X$ and for all objects X, Y and Z, and morphisms $f : X \to Y$ from X to Y and $g : Y \to Z$ from Y to Z their *composition* $g \circ f : X \to Z$ is a morphism in $\mathcal{C}$ from X to Z. The identities and compositions are required to satisfy the identity law that $f \circ \mathsf{id}_X = f = \mathsf{id}_Y \circ f$ for all morphism $f : X \to Y$ and the associativity law that $(h \circ g) \circ f = h \circ (g \circ f)$ for all $f : X \to Y$, $g : Y \to Z$ and $h : Z \to U$. For a morphism $f : X \to Y$ in $\mathcal{C}$ we call the object X the *domain* of f and the object Y the *codomain* of f.

Examples of categories, which we use in this paper, are the category of sets Set, which has all the sets as its objects and functions between them as morphism, and the category of uncertainty spaces Unc, which has uncertainty spaces as objects and measurable functions as morphisms. These clearly satisfy the identity and associativity laws if one uses the standard identity functions as identity morphisms and the standard compositions of functions for the composition. A different kind of example is the category CS of all choice structures as defined in Section 3 with morphisms of choice structures between them. The results of this paper mostly concern the structure of this category CS.

A more technical example of a category is Unc^2, which has pairs $X = (X_0, X_1)$ of uncertainty spaces X_0 and X_1 as objects and in which the morphisms from $X = (X_0, X_1)$ to $Y = (Y_0, Y_1)$ are pairs of measurable functions $\varphi = (\varphi_0, \varphi_1)$. One can check that this is a category, when one gives the obvious component-wise definition of identities and composition. We use the category Unc^2 as a convenient abstraction for the technical results of this appendix because it concisely represents situations that concern the uncertainty of two players. In an object $X = (X_0, X_1)$ of Unc^2, the uncertainty space X_0 represents the uncertainty of Ann, whereas X_1 represents the uncertainty of Bob.

A.2 Functors

A *functor* G from a category $\mathcal{C}$ to a category $\mathcal{D}$ is an assignment of an object GX in $\mathcal{D}$ to every object X of $\mathcal{C}$ and an assignment of a morphism $Gf : GX \to GY$ in $\mathcal{D}$ to every morphism $f : X \to Y$ in $\mathcal{C}$ that preserves identities and compositions, meaning that $G\mathsf{id}_X = \mathsf{id}_{GX}$ for all X and $G(g \circ f) = Gg \circ Gf$ for all g and f. If G is a functor from a category $\mathcal{C}$ to the same category $\mathcal{C}$ one also calls G an *endofunctor*.

A *contravariant functor* from $\mathcal{C}$ to $\mathcal{D}$ is a functor from $\mathcal{C}$ to $\mathcal{D}^{\mathsf{op}}$, where $\mathcal{D}^{\mathsf{op}}$ is just like $\mathcal{D}$, but the direction of all morphisms is turned around. When G is a contravariant functor from $\mathcal{C}$ to $\mathcal{D}$, one usually just thinks of it as turning morphisms around, in the sense that $Gf : GY \to GX$ for all $f : X \to Y$. If one wants to emphasize that a functor G from $\mathcal{C}$ to $\mathcal{D}$ is not contravariant, one calls G *covariant*.

An example of a functor from this paper is the mapping sending an uncertainty space X to the set FX of acts for X. This is a contravariant functor from uncertainty spaces to sets, that in category theory is called the contravariant hom-functor $\mathrm{Hom}(-, Z)$, where Z is the finite space of outcomes.

Another example of a functor is the mapping $\mathcal{C}$ from Section 2.3, which sends a set X to the uncertainty space $\mathcal{C}X$ of choice functions over that set. This defines a contravariant functor from the category of sets to the the category of uncertainty spaces. It is straightforward to check that it preserves identities. To see that $\mathcal{C}$ preserves the composition of morphisms one sees after unfolding the definitions that this amounts to checking the following equality:

$$f^{-1}[g^{-1}[C(g[f[K]])]] \cap K = f^{-1}[g^{-1}[C(f[g[K]])] \cap g[K]] \cap K,$$

for all finite sets $K \subseteq Y$, and functions $f : X \to Y$ and $g : Y \to W$.

The mapping Γ from Section 2.4 is a covariant endofunctor on the category of uncertainty spaces that results from first applying the contravariant functor F from uncertainty spaces to sets and then the contravariant functor $\mathcal{C}$ to get back to uncertainty spaces. Because the effects of the two contravariant functors F and $\mathcal{C}$ cancel out, this means that Γ is a covariant functor.

The mappings $\mathcal{P}$ and Π from [DT08], which we survey in Section 4.1, can also be seen as functors analogous to $\mathcal{C}$ and Γ. That is, $\mathcal{P}$ is a contravariant functor from Set to Unc and Π is the covariant endofunctor on Unc that results from first applying the contravariant functor F from Unc to Set and then the functor $\mathcal{P}$.

A.3 Coalgebras

We now explain how choice structures can be seen as an instance of the more general notion of a coalgebra for an endofunctor in some category.

To define the notion of a coalgebra we need to fix a category $\mathcal{C}$ and an endofunctor G on $\mathcal{C}$, that is, a functor G from $\mathcal{C}$ to $\mathcal{C}$. A *G-coalgebra* (X, ξ) is an object X of $\mathcal{C}$ together with a morphism $\xi : X \to GX$ in $\mathcal{C}$. The class of all G-coalgebras can be turned into a category by taking as morphism from a G-coalgebra (X, ξ) to a G-coalgebra (Y, υ) all morphisms $f : X \to Y$ in the category $\mathcal{C}$ that satisfy the property that $\upsilon \circ f = Gf \circ \xi$.

Choice structures for fixed sets S_a and S_b of strategies can be defined as coalgebras for a particular functor $\Gamma^{\heartsuit}$ over the category Unc^2. The functor $\Gamma^{\heartsuit}$ maps an object (X_1, X_2) to the object $(\Gamma(S_b \times X_2), \Gamma(S_a \times X_1))$. The change of indices in the components is intentional, as it corresponds to encoding attitudes about the opponent's attitudes. For morphisms the functor $\Gamma^{\heartsuit}$ sends a pair of measurable functions (φ_1, φ_2) from (X_1, X_2) to (Y_1, Y_2) to the pair of measurable functions $(\Gamma(\mathsf{id}_{S_b} \times \varphi_2), \Gamma(\mathsf{id}_{S_a} \times \varphi_1))$ from $(\Gamma(S_b \times X_2), \Gamma(S_a \times X_1))$ to $(\Gamma(S_b \times Y_2), \Gamma(S_a \times Y_1))$.

We can view every choice structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$, defined according to Definition 1, as a $\Gamma^{\heartsuit}$-coalgebra $((T_a, T_b), (\theta_a, \theta_b))$ on the object (T_a, T_b) of Unc^2 with the morphism $(\theta_a, \theta_b) : (T_a, T_b) \to \Gamma^{\heartsuit}(T_a, T_b)$. Conversely, every $\Gamma^{\heartsuit}$-coalgebra $((X_a, X_b), (\xi_a, \xi_b))$ contains measurable functions $\xi_a : X_a \to \Gamma(S_b \times X_b)$ and $\xi_b : X_b \to \Gamma(S_a \times X_a)$ and thus gives rise to a choice structure (X_a, X_b, ξ_a, ξ_b). One can also check that the morphisms of choice structures, as defined in Definition 1, correspond precisely to the coalgebra morphisms between $\Gamma^{\heartsuit}$-coalgebras.

Analogous to choice structures one can view preference structures as coalgebras for a functor. To this aim one considers the functor $\Pi^{\heartsuit}$ from Unc^2 to Unc^2 that is defined by replacing all occurrences of Γ in the definition of $\Gamma^{\heartsuit}$ with Π. This means that $\Pi^{\heartsuit}$ sends an object (X_1, X_2) to $(\Gamma(S_b \times X_2), \Gamma(S_a \times X_1))$ and a morphism (φ_1, φ_2) to $(\Pi(\mathsf{id}_{S_b} \times \varphi_2), \Pi(\mathsf{id}_{S_a} \times \varphi_1))$.

A.4 Isomorphisms and epic or monic morphisms

A morphism $f : X \to Y$ in a category $\mathcal{C}$ is an *isomorphism* if there exists a morphism $g : Y \to X$ such that $g \circ f = \mathsf{id}_X$ and $f \circ g = \mathsf{id}_Y$. One writes $X \simeq Y$ in case there exists an isomorphism $f : X \to Y$. One can check that in the category of sets the isomorphism are precisely the bijective functions and in the category Unc they are the bijective measurable functions whose inverse is also measurable, analogously to the homeomorphism between topological spaces. The isomorphisms in Unc^2 are those pairs (φ_1, φ_2) of morphism such that both φ_1 and φ_2 are isomorphisms in Unc.

A morphism $f : X \to Y$ is *epic* if for all further morphisms $g, h : Y \to T$ it holds that if $g \circ f = h \circ f$ then already $g = h$. A morphism $f : X \to Y$ is *monic* if for all further morphisms $g, h : T \to X$ it holds that if $f \circ g = f \circ h$ then already $g = h$. One can check that in the category of sets epic morphism are precisely the surjective functions and monic morphism are precisely the injective functions. Similarly, in the category of uncertainty spaces are the epic and monic morphisms are the surjective and injective measurable functions. Epic and monic morphisms in Unc^2 are epic and monic in both components. That is, (φ_1, φ_2) is epic if both φ_1 and φ_2 are epic, and analogously for monic.

A functor F from $\mathcal{C}$ to $\mathcal{D}$ *preserves* an epic morphism f of $\mathcal{C}$ if Ff is epic in $\mathcal{D}$, and F *preserves* a monic morphism f if Ff is monic. Note that, because an epic morphism in $\mathcal{D}^{\mathsf{op}}$ is monic in $\mathcal{D}$, a contravariant functor F from $\mathcal{C}$ to $\mathcal{D}$ preserves an epic morphism f if Ff is monic in $\mathcal{D}$, and analogously for the preservation of monic morphisms.

We show in Sections B.1 and B.2 that Γ preserves all epic morphisms and that it preserves all monic morphisms $\varphi : X \to Y$ for which Y is a discrete space.

A.5 Terminal objects, products and limits of chains

We make use of three different instances of the notion of a limit: terminal objects, products and limits of chains. We define these three concepts separately in the following, but to the interested reader it should be noted that they are all instances of the more general concept of a limit for a diagram, which is discussed in all of the texts on category theory cited above.

A *terminal object* in a category $\mathcal{C}$ is any object $\top$ with the property that for every object T of $\mathcal{C}$ there exists a unique morphism $!_T : T \to \top$. It follows directly from this definition that any two terminal objects in some category need to be isomorphic. Hence, as long as one only cares about the existence of objects up-to isomorphism, one can assume that there is a unique terminal object of a category, if any terminal object exists in the category.

The terminal object in the category Unc can be defined as the uncertainty space $\top = (\{\star\}, \{\emptyset, \{\star\}\})$ which contains just one point. The terminal object in the category Unc^2 can be defined componentwise as the object $(\top, \top)$ where $\top$ is the terminal object of Unc. A more interesting example of a terminal object is the universal choice structure. The universality property from Theorem 2 expresses precisely that the universal choice structure is the terminal object in the category of choice structures.

Given two objects X and Y of a category $\mathcal{C}$, a *product* of X and Y is an object of $\mathcal{C}$, written as $X \times Y$, together with two morphism $\pi_1 : X \times Y \to X$ and $\pi_2 : X \times Y \to Y$, which have the following universal property: For every object T and morphisms $f : T \to X$ and $g : T \to Y$ there is a morphism $u : T \to X \times Y$, often written as (f, g), that is unique for having the property that $f = \pi_1 \circ u$ and $g = \pi_2 \circ u$.

One can see that any two objects with this universal property, relative to fixed X and Y, need to be isomorphic. Hence, one often speaks of the product of X and Y as if it was unique.

In the category Unc one can define the product $X \times Y$ of two uncertainty spaces X and Y to be an uncertainty space whose states are all pairs (x, y) where x is a state of X and y is a state of Y. The measurable sets of states are generated by taking finite unions and complements of cylinders, that is, sets of the form $U \times Y$ and $X \times V$ for measurable U in X and V measurable in Y. It is clear that with this algebra the projections $\pi_1 : X \times Y \to X, (x, y) \mapsto x$ and $\pi_2 : X \times Y \to Y, (x, y) \mapsto y$ are measurable functions. Moreover, one can check that this definition of the product has the universal property that for each space T together with measurable functions $\varphi : T \to X$ and $\psi : T \to Y$ there is a unique measurable function $\mu : T \to X \times Y, t \mapsto (\varphi(t), \psi(t))$ such that $\varphi = \pi_1 \circ \mu$ and $\psi = \pi_2 \circ \mu$.

We need a further piece of notation related to the product: given $f : X \to Z$ and $g : Y \to U$ we write $f \times g : X \times Y \to Z \times U$ for the morphism $f \times g = (f \circ \pi_1, g \circ \pi_2)$. It is easy to check that this definition generalizes our earlier definition of $\varphi \times \psi$ for measurable function φ and ψ in the category of uncertainty spaces.

A crucial categorical notion for the construction of the universal choice structure is that of the limit of a countable cochain. A *countable cochain* $(X_n, f_n)_{n\in\omega}$ in a category $\mathcal{C}$ consists of an object X_n of $\mathcal{C}$ for every natural number $n \in \omega$ and a morphism $f_n : X_{n+1} \to X_n$ for every $n \in \omega$. The ξ_n are also called the *coherence morphism* of the cochain.

It is clear that we can compose the morphisms f_n to obtain a morphisms $f_n^m = f_n \circ \cdots \circ f_{m-1} : X_m \to X_n$ for all $n, m \in \omega$ with $n \leq m$, where $f_n^n = \mathsf{id}_{X_n} : X_n \to X_n$ is just the identity on X_n.

We call the countable cochain $(X_n, f_n)_{n\in\omega}$ *epic*, if all the morphisms f_n are epic in $\mathcal{C}$.

A *limit* of the countable cochain $(X_n, f_n)_{n\in\omega}$ is an object X_ω together with projection morphisms $p_n : X_\omega \to X_n$ such that $p_n = f_n \circ p_{n+1}$ for all $n \in \omega$ which satisfies the following universal property: For every object T together with morphisms $g_n : T \to X_n$, satisfying $g_n = f_n \circ g_{n+1}$ for all $n \in \omega$, there is a morphism $u : T \to X_\omega$ that is unique for having the property that $g_n = p_n \circ u$ for all $n \in \omega$. As for the product, and the terminal object, one can show, using this universal property, that any two limits of a given cochain are isomorphic.

For every cochain $(X_n, \xi_n)_{n\in\omega}$ in the category Unc we can define its limit X_ω to be the uncertainty space that has as its states all sequences $x = (x_0, x_1, \ldots, x_n, \ldots)$, where x_n is a state from X_n for all $n \in \omega$, which are coherent in the sense that $x_n = \xi_n(x_{n+1})$ for all $n \in \omega$. We can then consider the projections $\zeta_n : X_\omega \to X_n, x \mapsto x_n$ for each $n \in \omega$. The measurable sets of the limit X_ω are defined to be all sets of the form $(\zeta_n)^{-1}[E]$ for some $n \in \omega$ and measurable set E of X_n. It is clear that this definition turns the projections $\zeta_n : X_\omega \to X_n$ into measurable functions. Moreover, using that all ξ_n are measurable, one can show that the measurable sets defined in this way are closed under finite unions and intersections. To check that this definition of the limit has the universal property observe that for any measurable space T with measurable functions $\varphi_n : T \to X_n$, such that $\varphi_n = \xi_n \circ \varphi_{n+1}$ for each $n \in \omega$, we can define the unique morphism $\mu : T \to X_\omega$ such that $\mu(t) = (\varphi_0(t), \varphi_1(t), \ldots, \varphi_n(t), \ldots)$ for all $t \in T$.

One can also define the limit for every cochain $((X_{0,n}, X_{1,n}), (\xi_{0,n}, \xi_{1,n}))_{n\in\omega}$ in Unc^2. This can be done component-wise, meaning that for each of the components of the pairs representing objects and morphism in Unc^2 we can just use the construction for Unc, described in the previous paragraph. For instance the limit itself is just the object $(X_{0,\omega}, X_{1,\omega})$ such that $X_{0,\omega}$ and $X_{1,\omega}$ are the limits of $(X_{0,n}, \xi_{0,n})_{n\in\omega}$ and $(X_{1,n}, \xi_{1,n})_{n\in\omega}$ in Unc. It is not hard to see that this again has the required universal property.

A functor F from $\mathcal{C}$ to $\mathcal{D}$ *preserves* a limit X_ω of a countable cochain

$(X_n, f_n)_{n\in\omega}$ if FX_ω is the limit for the countable cochain $(FX_n, Ff_n)_{n\in\omega}$.

The main technical result in this appendix is that the functor Γ preserves limits of epic countable cochains. In the proof of this claim in Section B.3 we investigate the preservation of limits of cochains under the contravariant functors F from Unc to Set and $\mathcal{C}$ from Set to Unc. For this purpose we need to understand limits of cochains in $\mathsf{Set}^{\mathsf{op}}$. Because morphisms in $\mathsf{Set}^{\mathsf{op}}$ are just morphisms from Set with swapped domain and codomain we have that a countable cochain $(X_n, f_n)_{n\in\omega}$ in $\mathsf{Set}^{\mathsf{op}}$ is just what is called a countable chain in Set, meaning that the X_n are all sets and the $f_n : X_n \to X_{n+1}$ are functions that now go from X_n to X_{n+1}. Moreover the limit X_ω of this cochain in $\mathsf{Set}^{\mathsf{op}}$ is what is actually called a colimit of chain in Set. The colimit X_ω of a chain $(X_n, f_n)_{n\in\omega}$ in Set can be described concretely as the disjoint union of all the X_n modulo the equivalence relation that identifies an x_n from X_n with an x_m from X_m if there is some $k \geq n, m$ such that $f^n_k(x_n) = f^m_k(x_m)$. Clearly, we can then define inclusions $p^n_\omega : X_n \to X_\omega$ for all $n \in \omega$. The universal property of this colimit is just the same as the universal property of limit in $\mathsf{Set}^{\mathsf{op}}$ with all morphisms turned around: For every set T with functions $h_n : X_n \to T$ such that $h_n = h_{n+1} \circ f_n$ for each $n \in \omega$, there is a unique function $u : X_\omega \to T$ such that $h_n = u \circ p^n_\omega$ for all $n \in \omega$.

A.6 Natural transformations

Let F and G be functors that both go from a category $\mathcal{C}$ to a category $\mathcal{D}$. Then a *natural transformation* η from F to G is an assignment of a morphism $\eta_X : FX \to GX$ in $\mathcal{D}$ to every object X in $\mathcal{C}$ such that for all morphisms $f : X \to Y$ in $\mathcal{C}$ it holds that $Gf \circ \eta_X = \eta_Y \circ Ff$.

An example of natural transformation from this paper is the measurable function $\lambda_X : \Pi X \to \Gamma X$ that is defined for every uncertainty space X. Proposition 2, which is proven in Section D.3, shows that λ is a natural transformation from the functor Π to the functor Γ.

B Properties of Γ

In this part of the appendix we prove crucial properties of the functor Γ which are needed for the coalgebraic construction of the universal choice structure. In case the reader tries to get an overview of the construction of the universal choice structure, they might first skip through this section and then refer back as needed when reading Sections C and D.

B.1 Γ preserves epic morphisms

We show that Γ preserves epic morphisms, which in Unc are just surjective measurable functions. Because Γ is the composition of F and $\mathcal{C}$ it suffices to show that F maps surjective measurable functions to injective functions in Set and that $\mathcal{C}$ maps injective functions to surjective measurable functions. It is easy to check the former using the definition of F and the cancellation property of epic morphism. That $\mathcal{C}$ maps injective functions to surjective measurable functions is shown in the following lemma:

Lemma 1. If $f : X \to Y$ is injective then $\mathcal{C}f : \mathcal{C}Y \to \mathcal{C}X$ is surjective.

Proof. It is easy to see that as f is injective it holds for all $U \subseteq X$ that $f^{-1}[f[U]] = U$. To check that $\mathcal{C}f$ is surjective consider any $C \in \mathcal{C}X$. We have to find a $C' \in \mathcal{C}Y$ such that $\mathcal{C}f(C') = C$. Define C' such that $C'(K) = f[C(f^{-1}[K])]$ for all finite $K \subseteq Y$. The following equality then follows for all $L \subseteq X$:

$$\begin{aligned}\mathcal{C}f(C')(L) &= f^{-1}[C'(f[L])] \cap L \\ &= f^{-1}[f[C(f^{-1}[f[L]])]] \cap L \\ &= C(L) \cap L = C(L).\end{aligned}$$

□

B.2 Γ preserves monic morphisms to discrete spaces

We show that Γ preserves monic morphisms, if the codomain has the discrete algebra. We again split the problem into two steps. First we show that F maps every surjective measurable function with a discrete codomain to an injective functions and then we show that $\mathcal{C}$ maps injective functions to surjective measurable functions.

Lemma 2. If the measurable function $\varphi : X \to Y$ is injective and Y has the discrete algebra then the function $F\varphi : FY \to FX$ is surjective.

Proof. Pick any $f \in FX$. We want to find a $f' \in FY$ such that $f = f' \circ \varphi$. We define $f'(y) = f(x)$, if there is some $x \in X$ such that $\varphi(x) = y$, and $f'(y) = z'$ for an arbitrary $z' \in Z$ otherwise. This is well-defined because φ is injective. The function f' is trivially measurable because Y has the discrete algebra and $f = f' \circ \varphi$ by definition of f'. □

Lemma 3. If $f : X \to Y$ is surjective then $\mathcal{C}f : \mathcal{C}Y \to \mathcal{C}X$ is injective.

Proof. It is easy to see that as f is surjective it holds for all $U \subseteq Y$ that $f[f^{-1}[U]] = U$. To check that $\mathcal{C}f$ is injective consider any two $C, C' \in \mathcal{C}Y$ such that $\mathcal{C}f(C) = \mathcal{C}f(C')$. We have to show that then $C(K) = C'(K)$ for all finite $K \subseteq Y$. Using that $f[f^{-1}[U]] = U$ we can compute:

$$\begin{aligned}\mathcal{C}f(C)(f^{-1}[K]) &= f^{-1}[C(f[f^{-1}[K]])] \cap f^{-1}[K] \\ &= f^{-1}[C(K)] \cap f^{-1}[K] = f^{-1}[C(K) \cap K] = f^{-1}[C(K)]\end{aligned}$$

Similarly we obtain also for C' that $\mathcal{C}f(C')(f^{-1}[K]) = f^{-1}[C'(K)]$. From the assumption that $\mathcal{C}f(C) = \mathcal{C}f(C')$ it follows that $f^{-1}[C(K)] = f^{-1}[C'(K)]$. Because f is surjective it follows that $C(K) = C'(K)$. □

B.3 Γ preserves limits of epic countable cochains

In this section we prove the main technical result of this paper:

Theorem 4. Γ preserves limits of countable epic cochains. This means that whenever we have a limit X_ω, with projections $\zeta_n : X_\omega \to X_n$ for all $n \in \omega$, of a countable cochain $(X_n, \xi_n)_{n\in\omega}$ in which the coherence morphisms ξ_n are epic for all n, then ΓX_ω, with projections $\Gamma\zeta_n : \Gamma X_\omega \to \Gamma X_n$, satisfies the universal property of the limit of the cochain $(\Gamma X_n, \Gamma\xi_n)_{n\in\omega}$, i.e., for every uncertainty space T together with measurable functions $\varphi_n : T \to \Gamma X_n$ for all $n \in \omega$ with $\varphi_n = \Gamma\xi_n \circ \varphi_{n+1}$ there is a unique measurable function $\mu : T \to \Gamma X_\omega$ such that $\varphi_n = \Gamma\zeta_n \circ \mu$ for all $n \in \omega$.

For the proofs from later sections in this appendix we need the the following corollary of Theorem 4.

Corollary 1. $\Gamma^\heartsuit$ preserves limits of countable epic cochains.

We split the proof of Theorem 4 into two steps. First, we show that F turns limits of epic cochains in Unc into colimits of chains in Set. Second, we show that $\mathcal{C}$ turns colimits of chains Set into limits of cochains in Unc. It follows that Γ preserves limits of epic cochains because Γ is defined as the composition of F and $\mathcal{C}$.

We first need a technical lemma stating a property that also plays an important role in Proposition 1 of [DT08]:

Lemma 4. Consider a countable cochain $(X_n, \xi_n)_{n\in\omega}$, with coherence morphisms $\xi_n : X_{n+1} \to X_n$, and let X_ω, with projections $\zeta_n : X_\omega \to X_n$, be its limit. Then, for every act $f \in FX_\omega$ there is a natural number $n \in \omega$ such that for every natural number $m \geq n$ there is an act $f' \in FX_m$ such that $f = f' \circ \zeta_m$.

Proof. Because $f : X_\omega \to Z$ is measurable and Z is a finite set we have that for every $z \in Z$ the set $f^{-1}[\{z\}]$ is measurable in X_ω. By the definition of the algebra on X_ω this means that for every $z \in Z$ there is some $n_z \in \omega$ such that $f^{-1}[\{z\}] = \zeta_{n_z}^{-1}[E_z]$ for some measurable set E_z in X_{n_z}. Let n be the maximum of all n_z for $z \in Z$. This maximum is a natural number because Z is finite.

Consider now any $m \geq n$. We have that for every $z \in Z$ there is some measurable $E_z^m = (\xi_{n_z}^m)^{-1}[E_z] \subseteq X_m$ such that $f^{-1}[\{z\}] = \zeta_m^{-1}[(\xi_{n_z}^m)^{-1}[E_z]] = \zeta_m^{-1}[E_z^m]$. We can then argue that in the image of ζ_m the E_z essentially partition X_m. More precisely, we show that for every $x \in X_\omega$ and $z \in Z$ if $\zeta_m(x) \in E_z^m$ then $z = f(x)$. To see this consider arbitrary such $x \in X_\omega$ and $z \in Z$ with $\zeta_m(x) \in E_z^m$. Because $E_z^m = (\xi_{n_z}^m)^{-1}[E_z]$ it follows that $\zeta_{n_z}(x) = \xi_{n_z}^m(\zeta_m(x)) \in E_z$. And because E_z was chosen such that $\zeta_{n_z}^{-1}[E_z] = f^{-1}[\{z\}]$ it follows that $f(x) = z$.

We define the act $f' \in FX_m$ such that $f'(x) = z$ for some fixed z such that $x \in E_z^m$, if there is such a z, and $f'(x)$ is an arbitrary element of Z otherwise. The latter is always possible because Z is not empty. It then follows immediately from the argument in the previous paragraph that $f(x) = f' \circ \zeta_m(x)$ for all $x \in x_\omega$. □

Proposition 3. F preserves limits of countable epic cochains. That is, it maps limits of cochains of uncertainty spaces to colimits of chains of sets.

Proof. Consider a cochain $(X_n, \xi_n)_{n\in\omega}$ of uncertainty spaces and let X_ω, together with projections $\zeta_n : X_\omega \to X_n$, be its limit. We show that FX_ω, together with the inclusions $F\zeta_n : FX_n \to FX_\omega$ has the universal property of the colimit of the chain $(FX_n, F\xi_n)_{n\in\omega}$. For this purpose consider any set T together with functions $h_n : X_n \to T$ such that $h_n = h_{n+1} \circ F\xi_n$ for each $n \in \omega$. We need to define the function $u : FX_\omega \to T$ such that $h_n = u \circ F\zeta_n$ for all $n \in \omega$ and show that it is unique with that property.

To define $u(f)$ for some act $f \in FX_\omega$ we use Lemma 4. From this lemma it follows that there is some k and act $f' \in FX_k$ such that $f = f' \circ \zeta_k = F\zeta_k(f')$.

We then set $u(f) = h_k(f')$. To check that this satisfies $h_n(g) = u \circ F\zeta_n(g)$ for all $n \in \omega$ and $g \in FX_n$ consider the act $F\zeta_n(g) \in FX_\omega$. By definition of u we have that $u(F\zeta_n(g)) = h_k(f')$ for some $f' \in FX_k$ such that $F\zeta_n(g) = F\zeta_k(f')$. We need to show that $h_k(f') = h_n(g)$. Assume that $n \leq k$. This is without loss of generality because we are only using the completely symmetric fact that $F\zeta_n(g) = F\zeta_k(f')$. Because $\zeta_n = \xi_n^k \circ \zeta_k$, it follows from $F\zeta_n(g) = F\zeta_k(f')$ that $f' \circ \zeta_k = g \circ \zeta_n = g \circ \xi_n^k \circ \zeta_k$. We obtain that $f' = g \circ \xi_n^k = F\xi_n^k(g)$ because ζ_k is epic. From this we can then conclude that $h_k(f') = h_n(g)$ since $h_k = h_n \circ F\xi_n^k$.

That u is unique also follows from Lemma 4. The possible values of $u(f)$ are completely determined because $f = F\zeta_n(f')$ for some n and $f' \in X_n$ and we need to ensure that $h_n(f') = u \circ F\zeta_n(f')$. □

Lemma 5. Consider a chain $(X_n, \xi_n)_{n \in \omega}$ of sets and let X_ω, with inclusions $\iota_n : X_n \to X_\omega$ for all n, be its colimit. Then for each finite $K \subseteq X_\omega$ there is an $m \in \omega$ and a $K' \subseteq X_m$ such that $\iota_m[K'] = K$.

Proof. By the definition of the colimit of a countable chain we have that for every $k \in X_\omega$ there exists some $k' \in X_{m_k}$ such that $\iota_{m_k}(k') = k$. Let m be the maximum of the finitely many m_k for all $k \in K$ and let then $K' = \{\xi_m^{m_k}(k') \in X_m \mid k \in K\}$. It then holds that $\iota_m[K'] = K$ because $\iota_m \circ \xi_m^{m_k}(k') = \iota_{m_k}(k') = k$ for every $k \in K$. □

Proposition 4. $\mathcal{C}$ preserves colimits of countable chains. That is, it maps colimits of chains of sets onto limits of cochains of uncertainty spaces.

Proof. Consider a chain $(X_n, \xi_n)_{n \in \omega}$ of sets and let X_ω, with inclusions $\iota_n : X_n \to X_\omega$ for all n, be its colimit. We show that $\mathcal{C}X_\omega$, with the $\mathcal{C}\iota_n : \mathcal{C}X_\omega \to \mathcal{C}X_n$ as the projections, has the universal property of the limit of the cochain $(\mathcal{C}X_n, \mathcal{C}\xi_n)_{n \in \omega}$. Hence, consider any uncertainty space T together with measurable functions $\varphi_n : T \to \mathcal{C}X_n$, satisfying $\varphi_n = \mathcal{C}\xi_n \circ \varphi_{n+1}$, for all $n \in \omega$. We need to show that there is a unique measurable function $\mu : T \to \mathcal{C}X_\omega$ such that $\mathcal{C}\iota_n \circ \mu = \varphi_n$ for all $n \in \omega$.

We first describe how to define the choice function $\mu(t) \in \mathcal{C}X_\omega$ on a finite set $K \subseteq X_\omega$. Let $m \in \omega$ be the least number such that there exists a $K' \subseteq X_m$ such that $\iota_m[K'] = K$. Such a number exists because of Lemma 5. Also observe that the choice of m and K' only depends on K but not on t. This is a property which we exploit below to show that μ is measurable. We then set the value of the choice function $\mu(t)$ on K to be

$$\mu(t)(K) = \iota_m[\varphi_m(t)(K')], \tag{2}$$

where $\varphi_m(t)(K') \subseteq K'$ denotes the elements of K' selected by the choice function $\varphi_m(t) \in \mathcal{C}X_m$. This is well-defined because we have $\iota_m[\varphi_m(t)(K')] \subseteq \iota_m[K'] = K$

To see that μ is measurable it suffices to check that the preimage $\mu^{-1}[B_L^K] \subseteq T$ of a basic measurable set $B_L^K = \{C \in \mathcal{C}X_\omega \mid C(K) \subseteq L\}$ is measurable in T. Hence, fix finite K and L with $L \subseteq K$ and let $m \in \omega$ and K' be defined from K as explained above. We are now going to show that $\mu^{-1}[B_L^K] = \varphi_m^{-1}[B_{L'}^{K'}]$, where $L' = \iota_m^{-1}[L]$. Because $\varphi_m : T \to \Gamma X_m$ is assumed to be measurable, it then follows that $\mu^{-1}[B_L^K]$ is measurable too. By unfolding the definitions one sees that the claim that $\mu^{-1}[B_L^K] = \varphi_m^{-1}[B_{L'}^{K'}]$ is equivalent to the claim that for all $t \in T$

$$\mu(t)(K) \subseteq L \qquad \text{iff} \qquad \varphi_m(t)(K') \subseteq \iota_m^{-1}[L].$$

By the definition of $\mu(t)$ above we see that the left side of this equivalence is the same as $\iota_m[\varphi_m(t)(K')] \subseteq L$, which is clearly equivalent to $\varphi_m(t)(K') \subseteq \iota_m^{-1}[L]$.

We need to verify that $\varphi_n = \mathcal{C}\iota_n \circ \mu$ for all $n \in \omega$. This requires that for every $t \in T$ the choice functions $\varphi_n(t)$ and $\mathcal{C}\iota_n \circ \mu(t)$ are the same. Hence, they need to have the same value on every finite set $L' \subseteq X_n$.

First let $K = \iota_n[L']$ and recall the definitions to see that

$$(\mathcal{C}\iota_n \circ \mu(t))(L') = \iota_n^{-1}[\mu(t)(\iota_n[L'])] \cap L' = \iota_n^{-1}[\iota_m[\varphi_m(t)(K')]] \cap L', \qquad (3)$$

for the $m \in \omega$ and the finite set $K' \subseteq X_m$ that are defined from K as described above. Our choice of K' above ensures that $\iota_m[K'] = K = \iota_n[L']$. Hence for every $k' \in K'$ there is some $l' \in L'$ such that $\iota_m(k') = \iota_n(l')$ and conversely for every $l' \in L'$ there is a $k' \in K'$ such that $\iota_m(k') = \iota_n(l')$. By the definition of identity for elements in the colimit X_ω it follows that for each such pair $k' \in X_m$ and $l' \in X_n$ with $\iota_m(k') = \iota_n(l')$ there is some $j \in \omega$ such that $\xi_j^m(k') = \xi_j^n(l')$. Let i be the maximum of all those $j \in \omega$, which exists because there are finitely many pairs $(k', l') \in K' \times L'$. It then clearly holds that $\xi_i^m(k') = \xi_i^n(l')$, whenever $\iota_m(k') = \iota_n(l')$ for $(k', l') \in K' \times L'$. Define then $L \subseteq X_i$ to be the finite set

$$L = \xi_i^m[K'] = \xi_i^n[L'].$$

Because $\varphi_n = \mathcal{C}\xi_i^n \circ \varphi_i$ we obtain that

$$\varphi_n(t)(L') = (\xi_i^n)^{-1}[\varphi_i(t)(\xi_i^n[L'])] \cap L' = (\xi_i^n)^{-1}[\varphi_i(t)(L)] \cap L'.$$

Similarly, because $\varphi_m = \mathcal{C}\xi_i^m \circ \varphi_i$ we obtain

$$\varphi_m(t)(K') = (\xi_i^m)^{-1}[\varphi_m(t)(\xi_i^m[K'])] \cap K' = (\xi_i^m)^{-1}[\varphi_i(t)(L)] \cap K'.$$

To prove $\varphi_n(t)(L') = (\mathcal{C}\iota_n \circ \mu(t))(L')$ it thus suffices by (3) to show that

$$(\xi_i^n)^{-1}[U] \cap L' = \iota_n^{-1}[\iota_m[(\xi_i^m)^{-1}[U] \cap K']] \cap L' \qquad (4)$$

for the set $U = \varphi_i(t)(L) \subseteq L$.

For the left-to-right inclusion of (4) consider any $l' \in L'$ such that $\xi_i^n(l') \in U$. We need that $l' \in \iota_n^{-1}[\iota_m[(\xi_i^m)^{-1}[U] \cap K']]$. From the definition of L we get that there is then some $k' \in K'$ such that $\xi_i^m(k') = \xi_i^n(l')$. Hence $k' \in (\xi_i^m)^{-1}[U] \cap K'$ and $\iota_n(l') = \iota_m(k')$. The latter two directly entail that $l' \in \iota_n^{-1}[\iota_m[(\xi_i^m)^{-1}[U] \cap K']]$.

For the right-to-left inclusion of (4) consider any $l' \in L'$ such that $\iota_n(l') = \iota_m(k')$ for some $k' \in (\xi_i^m)^{-1}[U] \cap K'$. With the definition of i above it then follows from $\iota_n(l') = \iota_m(k')$ that $\xi_i^n(l') = \xi_i^m(k') \in U$. Hence, $l' \in (\xi_i^m)^{-1}[U]$ and we are done.

Lastly, we show that $\mu(t)$ is completely determined by the requirement that $\varphi_n(t) = \mathcal{C}\iota_n \circ \mu(t)$ for all $n \in \omega$. Consider any $\nu : T \to \mathcal{C}X_\omega$ such that $\varphi_n(t) = \mathcal{C}\iota_n \circ \nu(t)$ for all $n \in \omega$. We argue that then $\nu(t)(K) = \mu(t)(K)$ for the μ defined as above and all finite $K \subseteq X_\omega$. Let $m \in \omega$ and $K' \subseteq X_m$ as in the definition of μ above, i.e., such that $\iota_m[K'] = K$. The requirement that $\varphi_m(t) = \mathcal{C}\iota_m \circ \nu(t)$ means that

$$\varphi_m(t)(K') = \iota_m^{-1}[\nu(t)(\iota_m[K'])] \cap K' = \iota_m^{-1}[\nu(t)(K)] \cap K'.$$

One can see that this entails that

$$\iota_m[\varphi_m(t)(K')] = \nu(t)(K).$$

The left-to-right inclusion is trivial and the other inclusion follows because $\nu(t)(K) \subseteq K$ and $\iota_m[K'] = K$. Therefore, we have shown that $\nu(t)(K)$ equals the expression that we use in (2) to define $\mu(t)(K)$. □

As an immediate consequence of the previous two propositions we obtain the main result of this section.

C Proofs for Section 3

Theorems 1 and 2 follow from the results about Γ together with well-known results [Wor99] about the final coalgebra. In the following, we sketch how these results from the general theory of coalgebra apply to the setting of choice structures.

C.1 Preliminary observations

Theorems 1 and 2 concern the the universal choice structure that is obtained from the choice hierarchies as described in Section 3.2. Let us first see how this fits into the coalgebraic set-up.

One can view the two uncertainty spaces $\Omega_{a,n}$ and $\Omega_{b,n}$ on the n-th level in the choice hierarchy as defining an object $\Omega_n = (\Omega_{a,n}, \Omega_{b,n})$ in the category Unc^2. In fact one can define these objects directly, just using the functor $\Gamma^{\heartsuit}$ as follows: We start with $\Omega_0 = \top$, where $\top$ is the terminal object in Unc^2, and then define inductively $\Omega_{n+1} = \Gamma^{\heartsuit}\Omega_n$. It is easy to see that with the exception of the 0-th level, which is omitted from the discussion in the main text, this yields the same sequence of pairs of uncertainty spaces as defined in Section 3.2. Similarly, the coherence morphism $\xi_{a,n}$ and $\xi_{b,n}$ can also be defined directly in Unc^2 with $\xi_0 = !_{\Gamma^{\heartsuit}\top} : \Gamma^{\heartsuit}\top \to \top$ and inductively $\xi_{n+1} = \Gamma^{\heartsuit}\xi_n : \Gamma^{\heartsuit}\Omega_{n+1} \to \Gamma^{\heartsuit}\Omega_n$. Thus, the sequence $(\Omega_n, \xi_n)_{n\in\Omega}$ forms a countable cochain in Unc^2.

The definition of the uncertainty spaces Ω_a and Ω_b of types in the the universal choice structures from Section 3.2 is such that these are precisely the limits of the countable cochains $(\Omega_{a,n}, \xi_{a,n})_{n\in\omega}$ and $(\Omega_{b,n}, \xi_{b,n})_{n\in\omega}$ in Unc. Because limits in Unc^2 are computed component-wise it follows that the object $\Omega = (\Omega_a, \Omega_b)$ in Unc^2 is the limit of the countable cochain $(\Omega_n, \xi_n)_{n\in\omega}$. Also recall that the limit comes with projections $\zeta_n = (\zeta_{a,n}, \zeta_{b,n}) : \Omega \to \Omega_n$ back into the chain such that $\zeta_n = \xi_n \circ \zeta_{n+1}$.

The crucial observation behind Theorems 1 and 2 is then that the functor $\Gamma^{\heartsuit}$ on Unc^2 preserves the limit of the epic cochain (Ω_n, ξ_n). This follows from Corollary 1, which states that $\Gamma^{\heartsuit}$ preserves limits of epic countable cochains. To apply Corollary 1 to the cochain $(\Omega_n, \xi_n)_{n\in\omega}$ we need to check that the coherence morphisms ξ_n for all $n \in \omega$ are epic. This can be checked directly by an induction over the definition of the ξ_n. In the base step $\xi_0 = (\xi_{a,0}, \xi_{b,0})$, and

$\xi_{a,0} = \Gamma\pi_1$, where π_1 is the projection out of a product. Because this projection is epic, and by the argument in Section B.1 Γ preserves epic morphisms, it follows that $\xi_{a,0}$ is epic. We reason analogously for $\xi_{b,0}$. That $\xi_{n+1} = (\xi_{a,n+1}, \xi_{b,n+1}) = (\Gamma(\mathsf{id}_{S_b} \times \xi_{b,n}), \Gamma(\mathsf{id}_{S_a} \times \xi_{a,n}))$ is epic, also follows easily because Γ preserves epic morphisms.

C.2 Theorem 1

Next consider the object $\Gamma^\heartsuit\Omega$ of Unc^2. For this object we can define morphisms into the cochain $(\Omega_n, \xi_n)_{n\in\omega}$ by setting $\tau_0 = !_{\Gamma^\heartsuit\Omega} : \Gamma^\heartsuit\Omega \to \Omega_0$ and $\tau_{n+1} = \Gamma^\heartsuit\zeta_n : \Gamma^\heartsuit\Omega \to \Gamma^\heartsuit\Omega_n$, which satisfy $\tau_n = \xi_n \circ \tau_{n+1}$ for all n. From Theorem 4 it follows that $\Gamma^\heartsuit\Omega$ together with the projections $\Gamma^\heartsuit\zeta_n = \tau_{n+1}$ is a colimit of the cochain $(\Gamma^\heartsuit\Omega_n, \Gamma^\heartsuit\xi_n)_{n\in\omega}$, which by definition is the same as the chain $(\Omega_{n+1}, \xi_{n+1})_{n\in\omega}$. We can use this observation to show that $\Gamma^\heartsuit\Omega$ together with the τ_n also satisfies the universal property of the limit of the cochain $(\Omega_n, \xi_n)_{n\in\omega}$. To this aim take any further object T of Unc^2 together with morphisms $g_n : T \to \Omega_n$ such that $g_n = \xi_n \circ g_{n+1}$ for all $n \in \omega$. We now just consider this g_n without g_0 as morphism into the chain $(\Gamma^\heartsuit\Omega_n, \Gamma^\heartsuit\xi_n)_{n\in\omega}$ satisfying that $g_{n+1} = \xi_{n+1} \circ g_{n+2} = \Gamma^\heartsuit\xi_n \circ g_{n+1}$ for all $n \in \omega$. Because $\Gamma^\heartsuit\Omega$ is a limit of this chain there must be then a unique morphism $u : T \to \Gamma^\heartsuit\Omega$ such that $g_{n+1} = \Gamma^\heartsuit\zeta_n \circ u = \tau_{n+1} \circ u$ for all $n \in \omega$. Additionally, we have that $g_0 = \tau_0 \circ u$ holds trivially because on both sides of the equation we have a morphism into the terminal object $\top = \Omega_0$ of Unc^2.

We have now seen that both Ω and $\Gamma^\heartsuit\Omega$ satisfy the universal property of limit for the cochain $(\Omega_n, \xi_n)_{n\in\omega}$. Theorem 1 then follows because there can be only one such object up to isomorphism.

More precisely, the isomorphism is given by the unique morphism $\mu = (\mu_a, \mu_b) : \Omega \to \Gamma^\heartsuit\Omega$ such that $\zeta_n = \tau_n \circ \mu$ for all $n \in \omega$, which exists because Ω is a limit. If one considers the components of this morphism one obtains the isomorphism $\mu_a : \Omega_a \to \Gamma(S_b \times \Omega_b)$ and $\mu_b : \Omega_b \to \Gamma(S_a \times \Omega_a)$ that are referred to in the formulation of Theorem 1.

C.3 Theorem 2

We now sketch the proof of Theorem 2. Consider an arbitrary choice structure $\mathcal{X} = (T_a, T_b, \theta_a, \theta_b)$, presented as a coalgebra $(T, \theta) = ((T_a, T_b), (\theta_a, \theta_b))$ for $\Gamma^\heartsuit$. We need to show that there is a unique morphism υ from (T, θ) to the universal choice structure (Ω, μ).

To prove the existence of υ first consider the measurable functions $\upsilon_0 = !_T : T \to \Omega_0$ and the inductively defined $\upsilon_{n+1} = \Gamma^\heartsuit\upsilon_n \circ \theta : T \to \Omega_{n+1}$. By an induction over n we can show that these measurable functions satisfy $\upsilon_n = \xi_n \circ \upsilon_{n+1}$. In the base case this is clear because there is only one morphism from T to the terminal object $\Omega_0 = \top$. For the inductive step we calculate as follows:

$$\upsilon_{n+1} = \Gamma^\heartsuit\upsilon_n \circ \theta = \Gamma^\heartsuit(\xi_n \circ \upsilon_{n+1}) = \Gamma^\heartsuit\xi_n \circ \Gamma^\heartsuit\upsilon_{n+1} \circ \theta = \xi_{n+1} \circ \upsilon_{n+2}.$$

Because Ω is defined as the limit of the sequence $(\Omega_n, \xi_n)_{n\in\Omega}$ it follows that there is a unique morphism $\upsilon : T \to \Omega$ with the property that for all $n \in \omega$

$$\upsilon_n = \zeta_n \circ \upsilon. \tag{5}$$

It remains to be seen that υ is a morphism of choice structures and that it is unique with this property.

To show that $\upsilon : T \to \Omega$ is a morphism from (T, θ) to (Ω, μ) we need to verify that $\mu \circ \upsilon = \Gamma^{\heartsuit}\upsilon \circ \theta$. We show this by proving that $\upsilon_n = \tau_n \circ \mu \circ \upsilon$ and $\upsilon_n = \tau_n \circ \Gamma^{\heartsuit}\upsilon \circ \theta$ hold for all $n \in \omega$. The equality $\mu \circ \upsilon = \Gamma^{\heartsuit}\upsilon \circ \theta$ then follows from the universal property for the limit $\Gamma^{\heartsuit}\Omega$ of $(\Omega_n, \xi_n)_{n\in\omega}$, with projections $\tau_n : \Gamma^{\heartsuit}\Omega \to \Omega_n$.

To see that $\upsilon_n = \tau_n \circ \mu \circ \upsilon$ we use that by its definition in Section C.2 μ satisfies $\tau_n \circ \mu = \zeta_n$. The claim then reduces to (5).

To see that $\upsilon_n = \tau_n \circ \Gamma^{\heartsuit}\upsilon \circ \theta$ we distinguish two cases. If $n = 0$ we have that both υ_0 and $\tau_0 \circ \Gamma^{\heartsuit}\upsilon \circ \theta$ are morphisms from T to the terminal object $\Omega_0 = \top$ of Unc^2. By the universal property of the terminal object they must be equal.

If n is strictly positive we can unfold the definition of υ_{n+1}, use (5) and then unfold the definition of τ_{n+1} to obtain the following computation

$$\upsilon_{n+1} = \Gamma^{\heartsuit}\upsilon_n \circ \theta = \Gamma^{\heartsuit}(\zeta_n \circ \upsilon) \circ \theta = \Gamma^{\heartsuit}\zeta_n \circ \Gamma^{\heartsuit}\upsilon \circ \theta = \tau_{n+1} \circ \Gamma^{\heartsuit}\upsilon \circ \theta.$$

To prove that υ is the only morphism of choice structures from (T, θ) to (Ω, μ) consider any other morphism $\upsilon' : T \to \Omega$ in Unc^2 such that $\mu \circ \upsilon' = \Gamma^{\heartsuit}\upsilon' \circ \theta$. We show that then $\upsilon_n = \zeta_n \circ \upsilon'$ for all $n \in \omega$, from which is follows that $\upsilon' = \upsilon$ because υ is defined as the unique morphism with the property (5). To prove $\upsilon_n = \zeta_n \circ \upsilon'$ we use an induction on n. In the base case we again have that υ_0 and $\zeta_0 \circ \upsilon'$ must be equal because they are both morphism from T to the terminal object $\Omega_0 = \top$ of Unc^2. In the inductive step we use the following computation:

$$\begin{aligned}
\upsilon_{n+1} &= \Gamma^{\heartsuit}\upsilon_n \circ \theta && \text{definition of } \upsilon_{n+1} \\
&= \Gamma^{\heartsuit}(\zeta_n \circ \upsilon') \circ \theta && \text{induction hypothesis} \\
&= \Gamma^{\heartsuit}\zeta_n \circ \Gamma^{\heartsuit}\upsilon' \circ \theta && \Gamma^{\heartsuit} \text{ functor} \\
&= \tau_{n+1} \circ \Gamma^{\heartsuit}\upsilon' \circ \theta && \text{definition of } \tau_{n+1} \\
&= \tau_{n+1} \circ \mu \circ \upsilon' && \text{assumption on } \upsilon' \\
&= \zeta_{n+1} \circ \upsilon' && \text{uniqueness property of } \mu
\end{aligned}$$

D Proofs for Section 4

D.1 Preliminary observations

It is easy to check that $\mathcal{P}$, and hence also Π and $\Pi^{\heartsuit}$ are functors, that is, they preserve identities and composition. The construction of the universal preference structure can then be carried out analogously to the construction for the universal choice structure given above. In fact it is only required to reprove a variant of Proposition 4 for the functor $\mathcal{P}$, which is done implicitly in the proofs of Section 3 from [DT08].

D.2 Proposition 1

We now argue that the map $m_X : \mathcal{P}X \to \mathcal{C}X$ is injective at every set X. To show this assume we have two preference relations $\preccurlyeq$ and $\preccurlyeq'$ over X such that $m_X(\preccurlyeq) = m_X(\preccurlyeq')$. We need to argue that $x \preccurlyeq x'$ iff $x \preccurlyeq' x'$ for all $x, x' \in X$.

Since the situation is symmetric it suffices to check one direction. Hence assume that $x \preccurlyeq x'$. We want to show that $x \preccurlyeq' x'$. It suffices to consider the case where $x \neq x'$ because otherwise $x \preccurlyeq' x'$ follows because $\preccurlyeq'$ is reflexive.

As $x \neq x'$ and $x \preccurlyeq x'$ it follows that it can not be the case that $x' \preccurlyeq x$, because otherwise there would be a contradiction with the anti-symmetry of $\preccurlyeq$. As we explain in Remark 1 this use of anti-symmetry is crucial. As $x \preccurlyeq x'$ and not $x' \preccurlyeq x$ it follows that x' is the only maximal element of the set $\{x, x'\}$ in the relation $\preccurlyeq$. Hence $m_X(\preccurlyeq)(\{x, x'\}) = \{x'\}$.

By the assumption that $m_X(\preccurlyeq) = m_X(\preccurlyeq')$ it follows that $m_X(\preccurlyeq')(\{x, x'\}) = \{x'\}$. But this is only possible if $x \preccurlyeq' x'$, which is what we had to show.

D.3 Proposition 2

Proposition 2 states that λ is a natural transformation from the functor Π to the functor Γ. It is easy to check that this reduces to the claim that m is a natural transformation from $\mathcal{P}$ to $\mathcal{C}$. This means that we need to show that for every function $f : X \to Y$ it holds that $m_X \circ \mathcal{P}f = \mathcal{C}f \circ m_Y$. Note that here the Y and X are swapped because $\mathcal{P}$ and $\mathcal{C}$ are contravariant functors.

Fix any function $f : X \to Y$, preference relation $\preccurlyeq \in \mathcal{P}Y$ and finite set $K \subseteq X$. We have to show that

$$m_X(\mathcal{P}f(\preccurlyeq))(K) = \mathcal{C}f(m_Y(\preccurlyeq))(K).$$

For the left-to-right inclusion take any $x \in m_X(\mathcal{P}f(\preccurlyeq))(K)$. We need to show that $x \in \mathcal{C}f(m_Y(\preccurlyeq))(K)$. This means we want to show that $x \in f^{-1}[m_Y(\preccurlyeq)(f[K])] \cap K$. Our assumption that $x \in m_X(\mathcal{P}f(\preccurlyeq))(K)$ means that x is a maximal element of the set K in the order $\preccurlyeq^f = \mathcal{P}f(\preccurlyeq)$. Hence $x \in K$ and it remains to show that $x \in f^{-1}[m_Y(\preccurlyeq)(f[K])]$, which means that $f(x)$ is a maximal element of the set $f[K]$ in the order $\preccurlyeq$. So consider any other element of $f[K]$, which must be of the form $f(x')$ for some $x' \in K$, and assume that $f(x) \preccurlyeq f(x')$. We need to show that then also $f(x') \preccurlyeq f(x)$. From equation (1) in Section 4.1 defining $\preccurlyeq^f$ it follows that $x \preccurlyeq^f x'$. Then we can use that x is a maximal element in $\preccurlyeq^f$ to conclude that $x' \preccurlyeq^f x$. Using (1) again we then obtain the required $f(x') \preccurlyeq f(x)$.

Now consider the right-to-left inclusion. Take any $x \in f^{-1}[m_Y(\preccurlyeq)(f[K])] \cap K$. We need to show that $x \in m_X(\mathcal{P}f(\preccurlyeq))(K)$, which means that x is a maximal element of the set K in the order $\preccurlyeq^f = \mathcal{P}f(\preccurlyeq)$. Clearly $x \in K$. To show that x is a $\preccurlyeq^f$-maximal element in K pick any other x' in K with $x \preccurlyeq x'$. We need to show that $x' \preccurlyeq x$. From $x \preccurlyeq x'$ it follows with (1) that $f(x) \preccurlyeq f(x')$. We now use that $x \in f^{-1}[m_Y(\preccurlyeq)(f[K])]$. From this it follows that $f(x) \in m_Y(\preccurlyeq)(f[K])$. This means that $f(x)$ is maximal in the set $f[K]$. Because $x' \in K$ it holds that also $f(x')$ is in the set $f[K]$. By the maximality of $f(x)$ in $f[K]$ it follows from $f(x) \preccurlyeq f(x')$ that $f(x') \preccurlyeq f(x)$. With (1) we obtain $x' \preccurlyeq x$.

D.4 Theorem 3

In the proof of Theorem 3 we need a further concept from category theory which is a generalization of monic morphism. A family of morphism $(f_j : X \to Y)_{j \in J}$ for any index set J is *jointly monic* if for all further morphisms $g, h : T \to Y$ it holds that if $f_j \circ g = f_j \circ h$ for all $j \in J$ then already $g = h$.

Families of monic morphism are closely related to limits of cochains. Using the universal property of the limit X_ω with projections ζ_n of a cochain $(X_n, f_n)_{n\in\omega}$ it is easy to show that the family of all projections $(p_n)_{n\in\omega}$ is jointly monic. Moreover, if we have another object T with a jointly monic family $(g_n : T \to X_n)_{n\in\omega}$ such that $g_n = f_n \circ g_{n+1}$ for all n then the unique morphism $u : T \to X_\omega$ that exists because of the universal property of the limit X_ω is monic.

To prove Theorem 3, let $\mathcal{U}' = (\Omega', \mu')$ be the universal preference structure, presented as coalgebra for $\Pi^\heartsuit$. We assume that Ω', μ', Ω'_n, ζ'_n, ... are defined in the same way as the objects Ω, μ, Ω_n, ζ_n, ... are defined in Section C for the universal choice structure, just using $\Pi^\heartsuit$ instead of $\Gamma^\heartsuit$.

Then consider the choice structure $\lambda(\mathcal{U}') = (\Omega', \lambda^\heartsuit_{\Omega'} \circ \mu')$ where $\lambda^\heartsuit$ is a natural transformation from $\Pi^\heartsuit$ to $\Gamma^\heartsuit$ that is defined such that it applies λ componentwise. Note that this definition of $\lambda(\mathcal{U}')$ corresponds to the one given in Section 4.3. Let υ be the unique morphism from $\lambda(\mathcal{U}')$ to the universal choice structure $\mathcal{U}$ that exists according to Theorem 2.

The claim of Theorem 3 is that this υ is monic. Because in Theorem 2 υ was obtain using the universal property of the limit Ω from the family of approximations $(\upsilon_n : \Omega'_n \to \Omega_n)_{n\in\omega}$ it suffices to show that this family is jointly monic.

Define morphisms $\delta_0 = !_{\Omega'_0} : \Omega'_0 \to \Omega_0$ and inductively $\delta_{n+1} = \Gamma^\heartsuit\delta_n \circ \lambda^\heartsuit_{\Omega'_n} : \Omega'_{n+1} \to \Omega_{n+1}$. One can show with an induction over n that all these δ_n are monic. The base case this holds because Ω'_0 is the terminal object of Unc^2 and in the inductive step we use Proposition 1 and the fact hat $\Gamma^\heartsuit$ preserves monics, which we show in Section B.2. The latter needs that the image of the injective measurable function $\delta_n : \Omega'_n \to \Omega_n$ that is preserved has the discrete algebra. This is the case because one can show that if S_a and S_b are finite then so are all the Ω'_n.

We then prove by induction on n that

$$\upsilon_n = \delta_n \circ \zeta'_n. \tag{6}$$

It follows that the υ_n are jointly monic because the ζ'_n are projections out of the limit Ω' and hence jointly monic and the δ_n are all monic.

For the base case, of (6), we have that $\upsilon_0 = \delta_0 \circ \zeta'_0$ because both morphism map to the terminal object Ω_0 of Unc^2.

For the inductive step we use the following computation:

$$\begin{aligned}
\upsilon_{n+1} &= \Gamma^\heartsuit\upsilon_n \circ \lambda^\heartsuit_{\Omega'} \circ \mu' && \text{definition of } \upsilon_{n+1} \\
&= \Gamma^\heartsuit(\delta_n \circ \zeta'_n) \circ \lambda^\heartsuit_{\Omega'} \circ \mu' && \text{induction hypothesis} \\
&= \Gamma^\heartsuit\delta_n \circ \Gamma^\heartsuit\zeta'_n \circ \lambda^\heartsuit_{\Omega'} \circ \mu' && \Gamma^\heartsuit \text{ functor} \\
&= \Gamma^\heartsuit\delta_n \circ \lambda^\heartsuit_{\Omega'_n} \circ \Pi^\heartsuit\zeta'_n \circ \mu' && \lambda^\heartsuit \text{ natural transformation} \\
&= \Gamma^\heartsuit\delta_n \circ \lambda^\heartsuit_{\Omega'_n} \circ \tau'_{n+1} \circ \mu' && \text{definition of } \tau'_{n+1} \\
&= \Gamma^\heartsuit\delta_n \circ \lambda^\heartsuit_{\Omega'_n} \circ \zeta'_{n+1} && \text{uniqueness property of } \mu' \\
&= \delta_{n+1} \circ \zeta'_{n+1} && \text{definition of } \delta'_{n+1}
\end{aligned}$$