A Focus System for the Alternation-Free μ-Calculus*

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Abstract. We introduce a cut-free sequent calculus for the alternation-free fragment of the modal μ-calculus. This system allows both for infinite and for finite, circular proofs and uses a simple focus mechanism to control the unravelling of fixpoints along infinite branches. We show that the proof system is sound and complete for the set of guarded valid formulas of the alternation-free μ-calculus.

Keywords: alternation-free mu-calculus · infinitary proof system · circular proof system · soundness · completeness

The modal μ-calculus $\mathcal{L}_\mu$, introduced in its present form by Kozen [16], is an extension of basic modal logic with least and greatest fixpoint operators. In the theory of formal program verification the formalism serves as a general specification language for describing properties of reactive systems, embedding many well-known logics such as LTL, CTL, CTL* and PDL. In fact, restricted to bisimulation-invariant properties, $\mathcal{L}_\mu$ has the same expressive power as monadic second-order logic [13], while it still has very reasonable computational properties, such as an EXPTIME-complete satisfiability problem [9]. Furthermore, the modal μ-calculus has many attractive logical properties, and interesting connections with for instance the theory of automata and infinite games. In particular, $\mathcal{L}_\mu$-formulas can be effectively represented as alternating tree automata, and vice versa [12, 26]. We refer to [4, 10, 5] for some surveys.

In this paper we contribute to the study of the modal μ-calculus by investigating one of its fragments. The theory of the full language is riddled with combinatorial intricacies involving the interaction between least- and greatest fixpoint operators. This interaction also lies at the root of the main drawback of the formalism, viz., that its formulas are not always easy to decipher. The *alternation-free μ-calculus* is the fragment $\mathcal{L}_\mu^{af}$ of $\mathcal{L}_\mu$ in which there is no real interaction between least and greatest fixpoint operators. This restriction comes with a decrease in expressive power, but many interesting logics, including LTL, CTL and PDL still embed into $\mathcal{L}_\mu^{af}$. Moreover, the expressive power of the full μ-calculus collapses to that $\mathcal{L}_\mu^{af}$ on some interesting classes of structures, such as

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transitive ones [2] or the ones with restricted connectivity [11]. The latter case generalises the particularly interesting example of the *linear time* μ-calculus [15]. Other reasons to study the alternation-free μ-calculus are that it corresponds in expressive power to a natural class of parity automata, viz., the ones with a so-called *weak* acceptance condition [19], and to the bisimulation-invariant fragment of the so-called *noetherian* variation of monadic second-order logic [6].

The problem that we address here is that of obtaining good proof systems for the alternation-free μ-calculus. Finding derivation systems for the full μ-calculus and proving their soundness and completeness is a notoriously difficult task, and successful applications of proof-theoretic techniques were few and far between for a long time. Kozen [16] introduced a natural axiomatisation for the full μ-calculus, and this system was proved to be complete by Walukiewicz [25]; Kozen's system, however, is a Hilbert-style axiomatisation. Niwiński & Walukiewicz [21] introduced some interesting tableau games, but these have a rather infinitary character. The same applies to the proof systems investigated by Dax et alii [7] and by Studer [24]. Fairly recently, however, Afshari & Leigh [1] obtained completeness of Kozen's axiomatisation using a series of cut-free *circular* derivation systems. A crucial ingredient for their results is an earlier proof system, developed by Jungteerapanich and Stirling [14, 23]. This system uses an intricate mechanism for annotating formulas to detect after finitely many steps when a branch of a proof may develop into a successful infinite branch in the sense of Niwínski & Walukiewicz' tableaux, thus obtaining a finite but circular proof.

In this paper we show that the approach of [14, 23] can be significantly simplified in the setting of the alternation-free μ-calculus. In our proof system it suffices to annotate formulas with just one bit of information, indicating whether a formula is *in focus* or not. This terminology is taken from the focus games for logics such as LTL and CTL by Lange & Stirling [17]. These are tableau-based games where every sequent of the tableau contains exactly one formula in focus; we generalise this so that a proof node may feature a *set* of formulas in focus. This focus mechanism is used to detect successful trails of fixpoint formulas in infinite branches of the proof (and seems to be unrelated to the literature on focused proof systems starting with [3]).

The bookkeeping of annotations in our system is very simple: as we follow the trail of a formula when moving up from the root in a Focus proof, we basically keep the annotation unchanged, with two exceptions. First, when we unfold a *least* fixpoint formula, we always drop the focus from its residual unfolding — whereas unfolding a *greatest* fixpoint formula has no influence on the annotations. And second, there are *focus change rules*, which put previously unfocused formulas into focus, or vice versa; their use however, is very restricted.

In this paper we introduce Focus_∞ and Focus as, respectively, an infinite and a finite but circular version of our focus proof system. We first show the equivalence of these two systems. Our main result concerns the soundness and completeness of Focus_∞; as an intermediate step in the proof we use a version of Niwiński & Walukiewicz' tableau games. Below we summarise the main line of argumentation in the paper (the number refers to the Theorem)

$$\vdash_{\mathsf{Focus}} \Phi \overset{1}{\iff} \vdash_{\mathsf{Focus}_\infty} \Phi \overset{5,6}{\iff} \Phi \in Win_{Prover}(\mathcal{G}(\mathbb{T})) \overset{4}{\iff} \Phi \text{ is valid.}$$

Here Φ denotes an arbitrary sequent of guarded alternation-free formulas.

Finally, although it may not be visible at the surface, our approach is heavily influenced by ideas from automata theory. Here we follow Jungteerapanich [14], whose annotations can be seen to encode a deterministic ω-automaton that recognises successful branches of infinite proofs. Where such an encoding in the setting of the full μ-calculus involves some version of the *Safra construction* [22], in the case of alternation-free formulas a much simpler mechanism suffices. Basically, our one-bit focus mechanism encodes the determisation procedure for weak ω-automata, as described in e.g. [8, Theorem 15.2.1].

Related versions More background and proof details can be found in our technical report [18].

1 Preliminaries

The modal μ-calculus The *formulas* of the language $\mathcal{L}_\mu$ of the modal μ-calculus are generated by the grammar

$$\varphi ::= p \mid \overline{p} \mid \bot \mid \top \mid (\varphi \vee \varphi) \mid (\varphi \wedge \varphi) \mid \Diamond\varphi \mid \Box\varphi \mid \mu x\, \varphi \mid \nu x\, \varphi,$$

where p and x are taken from a fixed set Prop of propositional variables and in formulas of the form $\mu x.\varphi$ and $\nu x.\varphi$ there are no occurrences of $\overline{x}$ in φ. It is well known that one can define a negation $\overline{\varphi} \in \mathcal{L}_\mu$ of any formula $\varphi \in \mathcal{L}_\mu$.

Formulas of the form $\mu x.\varphi$ ($\nu x.\varphi$) are called *μ-formulas* (*ν-formulas*, respectively); formulas of either kind are called *fixpoint formulas.* The operators μ and ν are called *fixpoint operators.* We use $\eta \in \{\mu, \nu\}$ to denote an arbitrary fixpoint operator and write $\overline{\eta} := \nu$ if $\eta = \mu$ and $\overline{\eta} = \mu$ if $\eta = \nu$. Formulas that are of the form $\Box\varphi$ or $\Diamond\varphi$ are called *modal.* Formulas of the form $\varphi \wedge \psi$ or $\varphi \vee \psi$ are called *boolean.* Formulas of the form p or $\overline{p}$ for some $p \in \mathsf{Prop}$ are called *literals* and the set of all literals is denoted by Lit; a formula is *atomic* if it is either a literal or an atomic constant, that is, $\top$ or $\bot$. We use standard notation and terminology for the binding of variables by the fixpoint operators and for substitutions. Given a fixpoint formula $\xi = \eta x.\chi$ we define its *unfolding* as the formula $\chi[\xi/x]$.

For every formula $\varphi \in \mathcal{L}_\mu$ we define the set $\mathsf{Clos}_0(\varphi)$ as follows

$$\begin{array}{llll}
\mathsf{Clos}_0(p) & := \varnothing & \mathsf{Clos}_0(\overline{p}) & := \varnothing \\
\mathsf{Clos}_0(\psi_0 \wedge \psi_1) & := \{\psi_0, \psi_1\} & \mathsf{Clos}_0(\psi_0 \vee \psi_1) & := \{\psi_0, \psi_1\} \\
\mathsf{Clos}_0(\Box\psi) & := \{\psi\} & \mathsf{Clos}_0(\Diamond\psi) & := \{\psi\} \\
\mathsf{Clos}_0(\mu x.\psi) & := \{\psi[\mu x.\psi/x]\} & \mathsf{Clos}_0(\nu x.\psi) & := \{\psi[\nu x.\psi/x]\}
\end{array}$$

If $\psi \in \mathsf{Clos}_0(\varphi)$ we call ψ a *residual* of φ and sometimes write $\varphi \to_C \psi$. We define the *closure* $\mathsf{Clos}(\varphi) \subseteq \mathcal{L}_\mu$ of φ as the least set Σ containing φ that is closed in the sense that $\mathsf{Clos}_0(\psi) \subseteq \Sigma$ for all $\psi \in \Sigma$. We define $\mathsf{Clos}(\Phi) = \bigcup_{\varphi \in \Phi} \mathsf{Clos}(\varphi)$ for any $\Phi \subseteq \mathcal{L}_\mu$. It is well known that $\mathsf{Clos}(\Phi)$ is finite iff Φ is finite. A *trace* is a sequence $(\varphi_n)_{n<\kappa}$, with $\kappa \leq \omega$, of formulas such that $\varphi_n \to_C \varphi_{n+1}$, for all n such that $n + 1 < \kappa$. If $\tau = (\varphi_n)_{n<\kappa}$ is an infinite trace, then there is a unique formula φ_τ that occurs infinitely often on τ and is a subformula of φ_n for cofinitely many n. This formula is always a fixpoint formula, and where it is of the form $\varphi_\tau = \eta x.\psi$ we call τ an *η-trace*.

A formula $\varphi \in \mathcal{L}_\mu$ is *guarded* if in every subformula $\eta x.\psi$ of φ all free occurrences of x in ψ are in the scope of a modality. It is well known that every formula can be transformed into an equivalent guarded formula, and one may verify that all formulas in the closure of a guarded formula are also guarded.

The semantics of the modal μ-calculus is given in terms of *(Kripke) models* $\mathbb{S} = (S, R, V)$, where S is a set whose elements are called *worlds* or *states*, $R \subseteq S \times S$ is a binary relation on S and $V : \mathsf{Prop} \to \mathcal{P}S$ is a function called the *valuation function*. The *meaning* $[\![\varphi]\!]^\mathbb{S} \subseteq S$ of a formula $\varphi \in \mathcal{L}_\mu$ relative to a model $\mathbb{S}$ is defined by induction on the complexity of φ:

$$
\begin{array}{llll}
[\![p]\!]^\mathbb{S} & := V(p) & [\![\overline{p}]\!]^\mathbb{S} & := S \setminus V(p) \\
[\![\bot]\!]^\mathbb{S} & := \varnothing & [\![\top]\!]^\mathbb{S} & := S \\
[\![\varphi \vee \psi]\!]^\mathbb{S} & := [\![\varphi]\!]^\mathbb{S} \cup [\![\psi]\!]^\mathbb{S} & [\![\varphi \wedge \psi]\!]^\mathbb{S} & := [\![\varphi]\!]^\mathbb{S} \cap [\![\psi]\!]^\mathbb{S} \\
[\![\Diamond\varphi]\!]^\mathbb{S} & := \{s \in S \mid R[s] \cap [\![\varphi]\!]^\mathbb{S} \neq \varnothing\} & [\![\Box\varphi]\!]^\mathbb{S} & := \{s \in S \mid R[s] \subseteq [\![\varphi]\!]^\mathbb{S}\} \\
[\![\mu x.\varphi]\!]^\mathbb{S} & := \bigcap\{U \subseteq S \mid [\![\varphi]\!]^{\mathbb{S}[x \mapsto U]} \subseteq U\} & [\![\nu x.\varphi]\!]^\mathbb{S} & := \bigcup\{U \subseteq S \mid [\![\varphi]\!]^{\mathbb{S}[x \mapsto U]} \supseteq U\}.
\end{array}
$$

Here, $\mathbb{S}[x \mapsto U]$ for some $U \subseteq S$ denotes the model (S, R, V'), where $V'(x) = U$ and $V'(p) = V(p)$ for all $p \in \mathsf{Prop}$ with $p \neq x$. We say that φ *is true* at s if $s \in [\![\varphi]\!]^\mathbb{S}$. A formula $\varphi \in \mathcal{L}_\mu$ is valid if $[\![\varphi]\!]^\mathbb{S} = S$ holds in all models $\mathbb{S}$ and two formulas $\varphi, \psi \in \mathcal{L}_\mu$ are *equivalent* if $[\![\varphi]\!]^\mathbb{S} = [\![\psi]\!]^\mathbb{S}$ for all models $\mathbb{S}$.

The alternation-free fragment Following the approach by Niwiński [20], we call a formula ξ alternation free if it satisfies the following: if ξ has a subformula $\eta x.\varphi$ then no free occurrence of x in φ can be in the scope of an $\overline{\eta}$-operator in φ. We let $\mathcal{L}_\mu^{af}$ denote the set of all alternation-free formulas. For an inductive definition of this set we refer to [18].

Example 1. For some examples of alternation-free formulas, observe that $\mathcal{L}_\mu^{af}$ contains all basic modal (i.e., fixpoint-free) formulas, as well as all $\mathcal{L}_\mu$-formulas that use μ-operators or ν-operators, but not both, and all modal and boolean combinations of such formulas. For a slightly more sophisticated example, consider the formula $\xi = \mu x.(\nu y.p \wedge \Box y) \wedge \Diamond x$. This formula does feature an alternating chain of fixpoint operators, in the sense that the ν-formula $\varphi = \nu y.p \wedge \Box y$ is a subformula of the μ-formula ξ. However, since the variable x does not occur in φ, this formula does belong to $\mathcal{L}_\mu^{af}$.

The language $\mathcal{L}_\mu^{af}$ is closed under taking respectively negations, unfoldings, subformulas and guarded equivalents of formulas. It follows from this that the closure operation restricts to alternation-free formulas. The next observation formulates an essential simplification of traces in the case of $\mathcal{L}_\mu^{af}$-formulas.

Proposition 1. *For any infinite trace $\tau = (\varphi_n)_{n<\omega}$ of $\mathcal{L}_\mu^{af}$-formulas the following are equivalent: (1) τ is an η-trace; (2) φ_n is an η-formula, for infinitely many n; (3) φ_n is an $\overline{\eta}$-formula, for at most finitely many n.*

2 The focus system

In this section we introduce our annotated proof system for the alternation-free μ-calculus. We consider two versions of the system, which we call Focus and Focus_∞, respectively. Focus_∞ is a proof system that allows proofs to be based on infinite, but finitely branching trees. The focus mechanism that is implemented by the annotations of formulas helps ensuring that all the infinite branches in a Focus_∞ proof are of the right shape. The proof system Focus can be seen as a finite variant of Focus_∞. The proof trees in this system are finite, but the system is circular in that it contains a discharge rule that allows to discharge a leaf of the tree if the same sequent is reached again closer to the root of the tree. As we will see, the two systems are equivalent in the sense that we may transform proofs in either variant into proofs of the other kind. We generally take a root-first perspective in proof search.

2.1 The proof systems Focus and Focus_∞

A *sequent* $(\Phi, \Psi, \ldots)$ is a finite set of guarded formulas, intuitively to be read *disjunctively.* We use standard notational conventions for sequents, e.g., we usually write $\varphi_1, \ldots, \varphi_i$ for the sequent $\{\varphi_1, \ldots, \varphi_i\}$, and $\varphi_1, \ldots, \varphi_i, \Phi$ for $\{\varphi_1, \ldots, \varphi_i\} \cup \Phi$. Given a sequent Φ we write $\Diamond\Phi$ for the sequent $\Diamond\Phi := \{\Diamond\varphi \mid \varphi \in \Phi\}$.

An *annotated formula* is a pair $(\varphi, a) \in \mathcal{L}_\mu^{af} \times \{f, u\}$; we usually write φ^a instead of (φ, a) and call a the *annotation* of φ. Given $a \in \{f, u\}$ we let $\overline{a}$ be its alternative, i.e., we define $\overline{u} := f$ and $\overline{f} := u$. Formulas annotated with f/u are said to be *in focus/out of focus*, respectively. A finite set of annotated formulas is called an *annotated sequent* $(\Sigma, \Gamma, \Delta, \ldots)$. In practice we will often be sloppy and refer to annotated sequents as sequents. Given a sequent Φ, we define $\Phi^a := \{\varphi^a \mid \varphi \in \Phi\}$. Conversely, we set $\widetilde{\Sigma} := \{\varphi \mid \varphi^a \in \Sigma, \text{ for some } a\}$. We abbreviate $\Sigma^f := \widetilde{\Sigma}^f$.

The proof rules of our focus proof systems Focus and Focus_∞ are given in Figure 1. We use standard terminology when talking about proof rules. Every (application of a) rule has one *conclusion* and a finite (possibly zero) number of *premises. Axioms* are rules without premises. The *principal* formula of a rule application is the formula in the conclusion to which the rule is applied. As non-obvious cases we have that all formulas are principal in the conclusion of the rule $\mathsf{R}_\Box$ and that the rule $\mathsf{D}^\times$ has no principal formula. In all cases other than the rule W the principal formula develops into one or more *residual* formulas in each of the premises. Principal and residual formulas are also called *active.*

Here are some more specific comments about the individual proof rules. The boolean rules ($\mathsf{R}_\wedge$ and $\mathsf{R}_\vee$) are fairly standard; observe that the annotation of the

$$\frac{}{p^a, \overline{p}^b}\ \mathsf{Ax1} \qquad \frac{}{\top^a}\ \mathsf{Ax2} \qquad \frac{\varphi^a, \psi^a, \Sigma}{(\varphi \vee \psi)^a, \Sigma}\ \mathsf{R}_\vee \qquad \frac{\varphi^a, \Sigma \qquad \psi^a, \Sigma}{(\varphi \wedge \psi)^a, \Sigma}\ \mathsf{R}_\wedge \qquad \frac{\varphi^a, \Sigma}{\Box\varphi^a, \Diamond\Sigma}\ \mathsf{R}_\Box$$

$$\frac{\varphi[\mu x.\varphi/x]^u, \Sigma}{\mu x.\varphi^a, \Sigma}\ \mathsf{R}_\mu \qquad \frac{\varphi[\nu x.\varphi/x]^a, \Sigma}{\nu x.\varphi^a, \Sigma}\ \mathsf{R}_\nu \qquad \frac{\Sigma}{\varphi^a, \Sigma}\ \mathsf{W} \qquad \frac{\varphi^f, \Sigma}{\varphi^u, \Sigma}\ \mathsf{F} \qquad \frac{\varphi^u, \Sigma}{\varphi^f, \Sigma}\ \mathsf{U}$$

$$\begin{array}{c} [\Sigma]^{\mathsf{x}} \\ \vdots \\ \dfrac{\Sigma}{\Sigma}\ \mathsf{D}^{\mathsf{x}} \end{array}$$

Fig. 1. Proof rules of the focus system

active formula is simply inherited by its subformulas. The fixpoint rules (R_μ and R_ν) simply unfold the fixpoint formulas; note, however, the difference between R_μ and R_ν when it comes to the annotations: in R_ν the annotation of the active ν-formula remains the same under unfolding, while in R_μ, the active μ-formula *loses focus* when it gets unfolded. The box rule $\mathsf{R}_\Box$ is the standard modal rule in one-sided sequent systems; the annotation of any formula in the consequent and its residual in the antecedent are the same.

The rule W is a standard *weakening rule*. Next to R_μ, the *focus rules* F and U are the only rules that change the annotations of formulas.[1] Finally, the *discharge rule* D is a special proof rule that allows us to discharge an assumption if it is repeating a sequent that occurs further down in the proof. Every application D^{x} of this rule is marked by a so-called *discharge token* x that is taken from some fixed infinite set $\mathcal{D} = \{\mathsf{x}, \mathsf{y}, \mathsf{z}, \dots\}$. In Figure 1 this is suggested by the notation $[\Sigma]^{\mathsf{x}}$. The precise conditions under which D^{x} can be employed are explained in Definition 1 below.

Definition 1. *A* pre-proof $\Pi = (T, P, \Sigma, \mathsf{R})$ *is a quadruple such that* (T, P) *is a, possibly infinite, tree with nodes* T *and parent relation* P *(with* Puv *meaning that* u *is the parent of* v*).* Σ *is a function that maps every node* $u \in T$ *to a non-empty annotated sequent* Σ_u*; and*

$$\mathsf{R} : T \to \{\mathsf{Ax1}, \mathsf{Ax2}, \mathsf{R}_\vee, \mathsf{R}_\wedge, \mathsf{R}_\Box, \mathsf{R}_\mu, \mathsf{R}_\nu, \mathsf{W}, \mathsf{F}, \mathsf{U}\} \cup \{\mathsf{D}^{\mathsf{x}} \mid \mathsf{x} \in \mathcal{D}\} \cup \mathcal{D} \cup \{\star\},$$

is a map that assigns to every node u *of* T *its* label $\mathsf{R}(u)$*, which is either (i) the name of a proof rule, (ii) a discharge token or (iii) the symbol* $\star$*.*

To qualify as a pre-proof, Π *is required to satisfy the following conditions:*

1. *If a node is labelled with the name of a proof rule then it has as many children as the proof rule has premises, and the annotated sequents at the node and its children match the specification of the proof rules in Figure 1.*

[1] The rule U is not really needed — in fact we prove completeness without it. We include U because of its convenience for constructing proofs.

2. *If a node is labelled with a discharge token or with $\star$ then it is a leaf. We call such nodes* non-axiomatic leaves *as opposed to the* axiomatic leaves *that are labelled with one of the axioms,* Ax1 *or* Ax2.
3. *For every leaf l that is labelled with a discharge token $\mathsf{x} \in \mathcal{D}$ there is exactly one node u in Π that is labelled with D^{x}. This node u, as well as its (unique) child, is a proper ancestor of l and satisfies $\Sigma_u = \Sigma_l$. In this situation we call l a* discharged leaf*, and u its* companion*; we write c for the function that maps a discharged leaf l to its companion $c(l)$.*
4. *If l is a discharged leaf with companion $c(l)$ then the path from $c(l)$ to l contains (4a) no application of the focus rules and (4b) at least one application of $\mathsf{R}_\Box$, while (4c) every node on this path features a formula in focus.*

Non-axiomatic leaves that are labelled with $\star$ (and thus not discharged), are called open*, as are the associated sequents. We call a pre-proof a* proof in Focus *if it is finite and does not have any open assumptions.*

A infinite branch $\beta = (v_n)_{n\in\omega}$ is successful *if there are infinitely many applications of $\mathsf{R}_\Box$ on β and there is some i such that for all $j \geq i$ the annotated sequent at v_j contains at least one formula that is in focus and none of the focus rules* F *and* U *is applied at v_j. A pre-proof is a* Focus_∞-proof *if it does not have any non-axiomatic leaves and all its infinite branches are successful.*

A plain sequent Φ is derivable *in* Focus*, notation: $\vdash_{\mathsf{Focus}} \Phi$, if there is a* Focus *proof for Φ^f; and similarly for* Focus_∞.

The idea behind the success condition on infinite branches (and the corresponding *path condition* 4 on finite Focus-proofs) is to force any infinite branch in a Focus_∞-proof (respectively, in the unravelling of a Focus-proof) to contain an infinite trace of formulas in focus. Since μ-formulas lose their focus when unfolded, such a trace then must be a ν-trace; and because of Proposition 1, every ν-trace will be of this form.

As an example of a Focus-proof consider the proof of the formula $\varphi \lor \psi$ in Figure 2, where $\varphi = \nu x.\Diamond(p \land x) \lor \Box(q \land x)$ and $\psi = \mu y.\Diamond((\overline{p} \land \overline{q}) \lor y)$. This example illustrates a crucial difference between our system and the ones from [17]. Whereas the sequents of [17] have exactly one formula in focus, it is crucial for us to allow for multiple formulas to be in focus at one single sequent. In the proof from Figure 2 both $\Diamond(p \land \varphi)$ and $\Box(q \land \varphi)$ need to be in focus at the sequent where $\mathsf{R}_\Box$ is applied. It is only above the application of $\mathsf{R}_\Box$, when the conjunction $\overline{p} \land \overline{q}$ is decomposed, that we know which of $p \land \varphi$ and $q \land \varphi$ needs to be in focus.

We close this section with a first observation about (pre-)proofs in this system. The (completely routine) proof is omitted.

Proposition 2. *Let $\Pi = (T, P, \Sigma, \mathsf{R})$ be some pre-proof with root r. Then all formulas occurring in Π belong to $\mathsf{Clos}(\widetilde{\Sigma_r})$.*

2.2 Circular and infinite proofs

We first show that Focus_∞ and Focus are the infinitary and circular version of the same proof system, and derive the same annotated sequents.

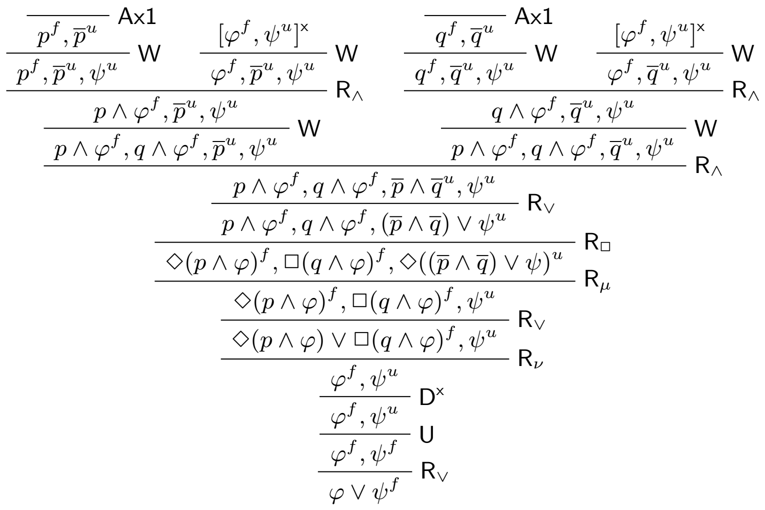

Fig. 2. A Focus-proof

Theorem 1. *Let Γ be an annotated sequent. Then Γ is provable in* Focus *iff it is provable in* Focus_∞*.*

Proof. (Sketch) The proof of the implication from left to right is based on a straightforward construction that (iteratively) *unravels* a given Focus-proof around its discharged leaves, creating a Focus_∞-proof in the limit.

For the opposite direction, fix a Focus_∞ pre-proof $\Pi = (T, P, \Sigma, \mathsf{R})$. If Π is finite we are done, so assume otherwise. A node u in Π is called a *successful repeat* if it has a proper ancestor t such that $\Sigma_t = \Sigma_u$, $\mathsf{R}(t) \neq \mathsf{D}$, and the path $[t, u]$ in Π satisfies condition 4 of Definition 1. It is then obvious by the definitions and Proposition 2 that every branch $\beta \in B^\infty$ contains a successful repeat. Define, for any $\tau \in B^\infty$, the number $\mathsf{l}(\tau) \in \omega$ as the least number $n \in \omega$ such that $\tau(n)$ is a successful repeat. This means that $\tau(\mathsf{l}(\tau))$ is the first successful repeat on τ. It is then possible to show, using König's Lemma, that the set

$$\widehat{Y} := \{\tau(\mathsf{l}(\tau)) \mid \tau \in B^\infty\}$$

is finite. Every element $l \in \widehat{Y}$ is a successful repeat; we may thus define a companion map $c : \widehat{Y} \to T$ by setting $c(l)$ to be the *first* ancestor t of l witnessing that l is a successful repeat. The map c takes care of the circular part of the finite tree (T', P') that will support the Focus-proof Π' of Γ. For a full and precise definition of Π' we have to add all ancestors of nodes in $\widehat{Y}$, and add a finite well-founded part, but this is not difficult. $\Box$

2.3 Thin and progressive proofs

When we prove the soundness of our proof system it will be convenient to work with (infinite) proofs that are in a certain normal form.

Definition 2. *An annotated sequent Σ is* thin *if there is no formula $\varphi \in \mathcal{L}_\mu^{af}$ such that $\varphi^f \in \Sigma$ and $\varphi^u \in \Sigma$. Given an annotated sequent Σ, we define its* thinning

$$\Sigma^- := \{\varphi^f \mid \varphi^f \in \Sigma\} \cup \{\varphi^u \mid \varphi^u \in \Sigma, \varphi^f \notin \Sigma\}.$$

A pre-proof $\Pi = (T, P, \Sigma, \mathsf{R})$ is thin *if for all $v \in T$ with $\varphi^f, \varphi^u \in \Sigma_v$ we have that $\mathsf{R}_v = \mathsf{W}$ and $\varphi^u \notin \Sigma_u$ for the unique u with Pvu.*

Note that one may obtain the thinning Σ^- from an annotated sequent Σ by removing the *unfocused* versions of the formulas with a double occurrence in Σ. Since $\Sigma^- \subseteq \Sigma$, one may derive Σ from Σ^- through a series of weakenings.

Definition 3. *An application of a boolean or fixpoint rule at a node u in a pre-proof $\Pi = (T, P, \Sigma, \mathsf{R})$ is* progressive *if for the principal formula $\varphi^a \in \Sigma_u$ it holds that $\varphi^a \notin \Sigma_v$ for all v with Puv.*[2] *Π itself is* progressive *if all applications of the boolean rules and the fixpoint rules in Π are progressive.*

Our main result here is the following.

Theorem 2. *Let Φ be some sequent. If Φ is derivable in* Focus *or* Focus_∞ *then it has a thin and progressive proof, both in* Focus *and in* Focus_∞*.*

3 Tableaux and tableau games

To prove soundness and completeness, as an intermediate step we use a (fairly straightforward) adaptation of Niwiński & Walukiewicz' tableau games [21].

Tableaux We first introduce tableaux, which are the graphs over which the tableau game is played. The nodes of a tableau for some sequent Φ are labelled with sequents consisting of formulas taken from the closure of Φ. Our system is based on the rules in Figure 3, where the tableau rules $\mathsf{Ax1}$, $\mathsf{Ax2}$, $\mathsf{R}_\vee$, $\mathsf{R}_\wedge$, R_μ and R_ν are direct counterparts of the focus proof rules with the same name.

The *modal rule* M can be seen as a game-theoretic version of the box rule $\mathsf{R}_\Box$ from the focus system, differing from it in two ways. First of all, the number of premises of M is not fixed, but depends on the number of box formulas in the conclusion; as a special case, if the conclusion contains no box formula at all, then the rule has an empty set of premises, similar to an axiom. Second, the rule M does allow side formulas in the consequent that are not modal; note however, that M has as its *side condition* (†) that this set Ψ contains atomic formulas only, and that it is *locally falsifiable*, i.e., Ψ does not contain $\top$ and there is no proposition letter p such that both p and $\overline{p}$ belong to Ψ. This side condition guarantees that M is only applicable if no other tableau rule is.

[2] Note that since we assume guardedness, the principal formula is different from its residuals.

$$\frac{}{p, \overline{p}, \Phi}\ \mathsf{Ax1} \qquad \frac{}{\top, \Phi}\ \mathsf{Ax2} \qquad \frac{\varphi, \psi, \Phi}{\varphi \vee \psi, \Phi}\ \mathsf{R}_\vee \qquad \frac{\varphi, \Phi \qquad \psi, \Phi}{\varphi \wedge \psi, \Phi}\ \mathsf{R}_\wedge$$

$$(\dagger)\ \frac{\varphi_1, \Phi \qquad \dots \qquad \varphi_n, \Phi}{\Psi, \Box\varphi_1, \dots, \Box\varphi_n, \Diamond\Phi}\ \mathsf{M} \qquad \frac{\varphi[\mu x.\varphi/x], \Phi}{\mu x.\varphi, \Phi}\ \mathsf{R}_\mu \qquad \frac{\varphi[\nu x.\varphi/x], \Phi}{\nu x.\varphi, \Phi}\ \mathsf{R}_\nu$$

Fig. 3. Rules of the tableau system

Definition 4. *A* tableau *is a quintuple* $\mathbb{T} = (V, E, \Phi, \mathsf{Q}, v_I)$*, where* (V, E) *is a directed graph,* $v_I \in V$ *is the* root *of the tableau,* Φ *maps every node* v *to a non-empty sequent* Φ_v*, and* $\mathsf{Q} : V \to \{\mathsf{Ax1}, \mathsf{Ax2}, \mathsf{R}_\vee, \mathsf{R}_\wedge, \mathsf{M}, \mathsf{R}_\mu, \mathsf{R}_\nu\}$ *associates a proof rule* Q_v *with each node* v *in* V*. Tableaux must satisfy the following:*

1. *If* $\mathsf{Q}(u) = \mathsf{R}$ *then the sequents at the node* u *and its successors match the specification of* R *as in Figure 3.*
2. *If* $\mathsf{Q}(u) = \mathsf{M}$ *then the side condition (*$\dagger$*) of* M *is met.*
3. *In any application of the rules* $\mathsf{R}_\vee, \mathsf{R}_\wedge, \mathsf{R}_\mu$ *and* R_ν*, the principal formula is not an element of the context* Φ*.*

A tableau $\mathbb{T}$ *is a* tableau for a sequent Φ *if* Φ *is the sequent of the root of* $\mathbb{T}$*.*

The following can easily be proved.

Proposition 3. *There is a tree-based tableau for every sequent* Φ*.*

Crucially, one needs to keep track of the development of individual formulas along infinite paths in a tableau. Fix a tableau $\mathbb{T} = (V, E, \Phi, \mathsf{Q}, v_I)$.

Definition 5. *For all nodes* $u, v \in V$ *such that* Euv *we define the* active trail relation $\mathsf{A}_{u,v} \subseteq \Phi_u \times \Phi_v$ *and the* passive trail relation $\mathsf{P}_{u,v} \subseteq \Phi_u \times \Phi_v$*, via the following case distinction:*

Case $\mathsf{Q}_u = \mathsf{R}_\vee$*: With* $\Phi_u = \{\varphi \vee \psi\} \uplus \Psi$ *and* $\Phi_v = \{\varphi, \psi\} \cup \Psi$*, we define* $\mathsf{A}_{u,v} = \{(\varphi \vee \psi, \varphi), (\varphi \vee \psi, \psi)\}$ *and* $\mathsf{P}_{u,v} = \Delta_\Psi$*, where* $\Delta_\Psi = \{(\varphi, \varphi) \mid \varphi \in \Psi\}$*.*

Case $\mathsf{Q}_u = \mathsf{R}_\wedge$*: With* $\Phi_u = \{\varphi_0 \wedge \varphi_1\} \uplus \Psi$ *and* v *corresponding to the conjunct* φ_i*, we set* $\mathsf{A}_{u,v} = \{(\varphi_0 \wedge \varphi_1, \varphi_i)\}$ *and* $\mathsf{P}_{u,v} = \Delta_\Psi$*.*

Case $\mathsf{Q}_u = \mathsf{R}_\eta$*: With* $\Phi_u = \{\eta x.\varphi\} \uplus \Psi$ *and* $\Phi_v = \{\varphi[\eta x.\varphi/x]\} \cup \Psi$*, we define* $\mathsf{A}_{u,v} = \{(\eta x.\varphi, \varphi[\eta x.\varphi/x])\}$ *and* $\mathsf{P}_{u,v} = \Delta_\Psi$*.*

Case $\mathsf{Q}_u = \mathsf{M}$*: With* $\Phi_u = \Psi \cup \{\Box\varphi_1, \dots, \Box\varphi_n\} \cup \Diamond\Phi$ *and* $\Phi_v = \{\varphi_v\} \cup \Phi$*, we define* $\mathsf{A}_{u,v} = \{(\Box\varphi_v, \varphi_v)\} \cup \{(\Diamond\varphi, \varphi) \mid \varphi \in \Phi\}$ *and* $\mathsf{P}_{u,v} = \varnothing$*.*

Finally, we define the general trail relation *as* $\mathsf{T}_{u,v} := \mathsf{A}_{u,v} \cup \mathsf{P}_{u,v}$*.*

Definition 6. *A* path *in* $\mathbb{T}$ *is simply a path in the underlying graph* (V, E) *of* $\mathbb{T}$*. A* trail *on such a path* $\pi = (v_n)_{n<\kappa}$ *is a sequence* $\tau = (\varphi_n)_{n<\kappa}$ *of formulas such that* $(\varphi_i, \varphi_{i+1}) \in \mathsf{T}_{v_i, v_{i+1}}$*, whenever* $i + 1 < \kappa$*. The* tightening $\widehat{\tau}$ *is obtained from* τ *by removing all* φ_{i+1} *from* τ *for which* $(\varphi_i, \varphi_{i+1})$ *belongs to the passive trail relation* $\mathsf{P}_{v_i, v_{i+1}}$*.*

Because of guardedness, any infinite path π in $\mathbb{T}$ witnesses infinitely many applications of the rule M; and for any trail $(\varphi_n)_{n<\omega}$ on π there are infinitely many i such that $(\varphi_i, \varphi_{i+1}) \in \mathsf{A}_{v_i,v_{i+1}}$. Furthermore, for any two nodes u, v with Euv and $(\varphi, \psi) \in \mathsf{T}_{u,v}$, we have either $(\varphi, \psi) \in \mathsf{A}_{u,v}$ and $\psi \in \mathsf{Clos}_0(\varphi)$, or $(\varphi, \psi) \in \mathsf{P}_{u,v}$ and $\varphi = \psi$. It is then not difficult to see that tightened trails are *traces*, and that the tightening of an infinite trail is infinite.

Definition 7. *Let $\tau = (\varphi_n)_{n<\omega}$ be an infinite trail on the path $\pi = (v_n)_{n<\omega}$ in some tableau $\mathbb{T}$. Then we call τ an η-*trail *if its tightening $\widehat{\tau}$ is an η-trace.*

Tableau games With each tableau $\mathbb{T}$ we associate a *tableau game* $\mathcal{G}(\mathbb{T})$, with two players, *Prover* (female) and *Refuter* (male).

Definition 8. *Given a tableau $\mathbb{T} = (V, E, \Phi, \mathsf{Q}, v_I)$, the* tableau game $\mathcal{G}(\mathbb{T})$ *is the (initialised) board game $\mathcal{G}(\mathbb{T}) = (V, E, O, \mathcal{M}_\nu, v_I)$ defined as follows. O is a partial map that assigns an* owner $O(v)$ *to some positions $v \in V$. Refuter owns all positions that are labelled with one of the axioms,* $\mathsf{Ax1}$ *or* $\mathsf{Ax2}$*, or with the rule* $\mathsf{R}_\wedge$*; Prover owns all position labelled with* M*; O is undefined on all other positions. In this context v_I will be called the* initial *position of the game.*

The set $\mathcal{M}_\nu$ is the winning condition *of the game (for Prover); it is defined as the set of infinite paths through the graph that carry a ν-trail.*

A *match* of the game consists of the two players moving a token from one position to another, starting at the initial position, and following the edge relation E. The owner of a position is responsible for moving the token from that position to an adjacent one (that is, an E-successor); in case this is impossible because the node has no E-successors, the player *gets stuck* and immediately loses the match. For instance, Refuter loses as soon as the token reaches an axiomatic leaf labelled $\mathsf{Ax1}$ or $\mathsf{Ax2}$; similarly, Prover loses at any modal node without successors. If the token reaches a position that is not owned by a player, that is, a node of $\mathbb{T}$ that is labelled with the proof rule $\mathsf{R}_\vee$, R_μ or R_ν, the token automatically moves to the unique successor of the position. If neither player gets stuck, the resulting match is infinite; we declare Prover to be its winner if the match, as an E-path, belongs to the set $\mathcal{M}_\nu$, that is, if it carries a ν-trail.

Finally, a *winning* strategy for a player P in $\mathcal{G}(\mathbb{T})$ is a way of playing that guarantees that P wins the resulting match, no matter how P's opponent plays.

Remark 1. If $\mathbb{T}$ is *tree-based* we may identify strategies for either player with *subtrees* S of $\mathbb{T}$ that contain the root of $\mathbb{T}$ and, for any node s in S, (1) contain exactly one successor of s in case the player owns the position s, and (2) contain all successors of s in case the player's opponent owns the position s.

The observations below are essentially due to Niwiński & Walukiewicz [21].

Theorem 3 (Determinacy). *Let $\mathbb{T}$ be a some tableau. Then precisely one of the players has a winning strategy in $\mathcal{G}(\mathbb{T})$.*

Theorem 4 (Adequacy). *Let $\mathbb{T}$ be a tableau for a sequent Φ. Then Refuter (Prover, respectively) has a winning strategy in $\mathcal{G}(\mathbb{T})$ iff the formula $\bigvee \Phi$ is refutable (valid, respectively).*

Corollary 1. *Let $\mathbb{T}$ and $\mathbb{T}'$ be two tableaux for the same sequent. Then Prover has a winning strategy in $\mathcal{G}(\mathbb{T})$ iff she has a winning strategy in $\mathcal{G}(\mathbb{T}')$.*

4 Soundness

In this section we establish the soundness of our system. Because of Theorem 4 and Theorem 1 it suffices to prove the following.

Theorem 5. *Let Φ be some sequent. If Φ is provable in Focus_∞ then there is some tableau $\mathbb{T}$ for Φ such that Prover has a winning strategy in $\mathcal{G}(\mathbb{T})$.*

We will prove the soundness theorem by transforming a thin and progressive Focus_∞-proof of Φ into a winning strategy for Prover in the tableau game associated with some tableau for Φ. We first adapt the notion of trail from tableaux to the setting of Focus_∞-proofs.

Definition 9. *Let $\Pi = (T, P, \Sigma, \mathsf{R})$ be a thin and progressive proof in Focus_∞. For all nodes $u, v \in V$ such that Puv we define the* active trail relation $\mathsf{A}_{u,v} \subseteq \Sigma_u \times \Sigma_v$ *and the* passive trail relation $\mathsf{P}_{u,v} \subseteq \Sigma_u \times \Sigma_v$, *via the following case distinction:*

Case $\mathsf{R}(u) = \mathsf{R}_\vee$: With $\Sigma_u = \{(\varphi \vee \psi)^a\} \uplus \Gamma$ and $\Sigma_v = \{\varphi^a, \psi^a\} \cup \Gamma$, we define $\mathsf{A}_{u,v} := \{((\varphi \vee \psi)^a, \varphi^a), ((\varphi \vee \psi)^a, \psi^a)\}$ and $\mathsf{P}_{u,v} := \Delta_\Gamma$.

In the cases where $\mathsf{R}(u) \in \{\mathsf{R}_\wedge, \mathsf{R}_\mu, \mathsf{R}_\nu, \mathsf{R}_\Box\}$ we proceed analogously.

Case $\mathsf{R}(u) = \mathsf{W}$: With $\Sigma_u = \Sigma_v \uplus \{\varphi^a\}$, we set $\mathsf{A}_{u,v} := \varnothing$ and $\mathsf{P}_{u,v} := \Delta_{\Sigma_v}$.

Case $\mathsf{R}(u) \in \{\mathsf{F}, \mathsf{U}\}$: With $\Sigma_u = \{\varphi^a\} \cup \Gamma$ and $\Sigma_v = \{\varphi^{\overline{a}}\} \cup \Gamma$, we define $\mathsf{A}_{u,v} = \varnothing$ and $\mathsf{P}_{u,v} = \{(\varphi^a, \varphi^{\overline{a}})\} \cup \Delta_\Gamma$.

We also define the general trail relation $\mathsf{T}_{u,v} := \mathsf{A}_{u,v} \cup \mathsf{P}_{u,v}$.

We inductively extend the trail relation $\mathsf{T}_{u,v}$ to any two nodes such that P^*uv by putting $\mathsf{T}_{u,u} := \Delta_{\Sigma_u}$, and if Puw and P^*wv then $\mathsf{T}_{u,v} := \mathsf{T}_{u,w} \,;\, \mathsf{T}_{w,v}$, where ; denotes relational composition.

As in the case of tableaux, we will be specifically interested in infinite trails and their tighentings. These are defined in exactly the same way as for tableaux.

The following observation concerns a central feature of our focus mechanism.

Proposition 4. *Every infinite branch in a thin and progressive Focus_∞-proof carries a ν-trail.*

Proof. Consider an infinite branch $\alpha = (v_n)_{n\in\omega}$ in some thin and progressive Focus_∞-proof $\Pi = (T, P, \Sigma, \mathsf{R})$. Then α is successful by assumption, so that we may fix a k such that for every $j \geq k$, the sequent Σ_{v_j} contains a formula in focus, and $\mathsf{R}(v_j)$ is not a focus rule. We claim that

$$\text{for every } j \geq k \text{ and } \psi^f \in \Sigma_{v_{j+1}} \text{ there is a } \chi^f \in \Sigma_{v_j} \text{ with } (\chi^f, \psi^f) \in \mathsf{T}_{v_j, v_{j+1}}. \tag{1}$$

To see this, let $j \geq k$ and $\psi^f \in \Sigma_{v_{j+1}}$. It is obvious that there is some annotated formula $\chi^a \in \Sigma_{v_j}$ with $(\chi^a, \psi^f) \in \mathsf{T}_{v_j,v_{j+1}}$. The key observation is now that in fact $a = f$, and this holds because the only way that we could have $(\chi^u, \psi^f) \in \mathsf{T}_{v_j,v_{j+1}}$ is if we applied the focus rule at v_j, which would contradict our assumption on the nodes v_j for $j \geq k$.

Now consider the graph (V, E) where

$$V := \{(j, \varphi) \mid k \leq j < \omega \text{ and } \varphi^f \in \Sigma_{v_j}\},$$
$$E := \{((j, \varphi), (j + 1, \psi)) \mid (\varphi^f, \psi^f) \in \mathsf{T}_{v_j,v_{j+1}}\}$$

This graph is directed, acyclic, infinite and finitely branching. Furthermore, it follows by (1) that every node (j, φ) is reachable in (V, E) from some node (k, ψ). Then by a (variation of) König's Lemma there is an infinite path $(n, \varphi^f_n)_{n\in\omega}$ in this graph. The induced sequence $\tau := (\varphi^f_n)_{n\in\omega}$ is a trail on α by definition of E. By the fact that α features infinitely many applications of $\mathsf{R}_\Box$, the tightening $\widehat{\tau}$ of τ must be infinite, and so τ is either a μ-trail or a ν-trail. But τ cannot feature infinitely many μ-formulas, simply because the rule R_μ attaches the label u to the unfolding of a μ-formula. This means that τ cannot be a μ-trail, and hence it must be a ν-trail. □

Proof of Theorem 5. Let $\Pi = (T, P, \Sigma, \mathsf{R})$ be a Focus_∞-proof for Φ^f. By Theorem 2 we may assume without loss of generality that Π is thin and progressive. We will construct a tableau $\mathbb{T} = (V, E, \Phi, \mathsf{Q}, v_I)$ and a winning strategy for Prover in $\mathcal{G}(\mathbb{T})$. Our construction will be such that (V, E) is a (generally infinite) tree, of which the winning strategy $S \subseteq V$ for Prover is a subtree, as in Remark 1.

In addition to the tableau $\mathbb{T}$ we define a function $g : S \to T$ satisfying the following three conditions, which will allow us to lift the ν-trails from Π to S:

1. If Euv then $P^* g(u) g(v)$.
2. The sequent $\Sigma_{g(u)}$ is thin, and $\widetilde{\Sigma}_{g(u)} \subseteq \Phi_u$.
3. If Euv and $(\psi^b, \varphi^a) \in \mathsf{T}^{\Pi}_{g(u),g(v)}$ then $(\psi, \varphi) \in \mathsf{T}^{\mathbb{T}}_{u,v}$.

The construction of $\mathbb{T}$, S and g is guided by the structure of Π and proceeds via an induction that starts from the root and in every step adds children to one of the nodes in the subtree S that is not yet an axiom. Nodes of $\mathbb{T}$ that are not in S are always immediately completely extended using Proposition 3, and thus need not be taken along in the inductive construction.

At step $n \in \omega$ of the construction, we are dealing with finite approximating objects $\mathbb{T}_n$, S_n and $g_n : S_n \to T$, and in the limit these will yield $\mathbb{T}$, S and g. Each $\mathbb{T}_n$ will be a *pre-tableau*, that is, an object as defined in Definition 4, except that we do not require the rule labelling to be defined for every leaf of the tree. The basic idea underlying the construction is that step n will take care of one such undetermined leaf of $\mathbb{T}_n$, say, l; the precise details of the construction (which are spelled out in [18]) depend on the nature of the proof rule applied in Π at the node $g_n(l)$.

It remains to be seen that S is a winning strategy for Prover in $\mathcal{G}(\mathbb{T})$. It is clear that she wins all finite matches that are played according to S because by construction all leaves in S are axioms. To show that she wins all infinite matches too, consider an infinite path $\beta = (v_n)_{n\in\omega}$ in S. We need to show that β contains a ν-trail. Using condition 1 it follows that there is an infinite path $\alpha = (t_n)_{n\in\omega}$ in Π such that for every $i \in \omega$ we have that $g(v_i) = t_{k_i}$ for some $k_i \in \omega$, and, moreover, $k_i \leq k_j$ if $i \leq j$. By Proposition 4 the infinite path α contains a ν-trail $\tau = \varphi_0^{a_0}\varphi_1^{a_1}\cdots$. With condition 3 it follows that $\tau' := \varphi_{k_0}\varphi_{k_1}\varphi_{k_2}\cdots$ is a trail on β. By Proposition 1, τ contains only finitely many μ-formulas; from this it is immediate that τ' also features at most finitely many μ-formulas. Thus, using Proposition 1 a second time, we find that τ' is a ν-trail, as required. □

5 Completeness

In this section we show that the focus systems are complete. Because of Theorem 4 and Theorem 1 it suffices to prove the following.

Theorem 6. *If Prover has a winning strategy in some tableau game for a sequent Φ then Φ is provable in Focus_∞.*

Proof. Let $\mathbb{T} = (V, E, \Phi, \mathsf{Q}, v_I)$ be a tableau for Φ and let S be a winning strategy for Prover in $\mathcal{G}(\mathbb{T})$. Because of Proposition 3, Corollary 1 and Remark 1, we may assume that $\mathbb{T}$ is tree based, with root v_I, and that $S \subseteq V$ is a subtree of $\mathbb{T}$. We will construct a Focus_∞-proof $\Pi = (T, P, \Sigma, \mathsf{R})$ for Φ^f.

Applications of the focus rules in Π will be very restricted. To start with, the unfocus rule U will not be used at all, and the focus rule F will only occur in the form of the following *total* focus rule F^t which is easily seen to be derivable as a series of successive applications of F:

$$\frac{\Phi^f}{\Phi^u}\ \mathsf{F}^t$$

We construct the pre-proof Π of Φ^f together with a function $g : S \to T$ in such a way that the following conditions are satisfied:

1. If Evu then $P^+g(v)g(u)$.
2. For every $v \in S$ and every infinite branch $\beta = (v_n)_{n\in\omega}$ in Π with $v_0 = g(v)$ there is some $i \in \omega$ and some $u \in S$ such that Evu and $g(u) = v_i$.
3. For every $\varphi \in \Phi_v$ there is a unique $a_\varphi \in \{f, u\}$ such that $\varphi^{a_\varphi} \in \Sigma_{g(v)}$. In particular, $\Sigma_{g(v)}$ is thin.
4. If Evu and $(\varphi, \psi) \in \mathsf{T}_{v,u}$ then $(\varphi^{a_\varphi}, \psi^{a_\psi}) \in \mathsf{T}_{g(v),g(u)}$.
5. If Evu, and s and t are nodes on the path from $g(v)$ to $g(u)$ such that P^+st, $(\chi^a, \varphi^f) \in \mathsf{T}_{g(v),s}$ for some $a \in \{f, u\}$ and $(\varphi^f, \psi^u) \in \mathsf{T}_{s,t}$, then $\chi = \varphi$ and χ is a μ-formula.
6. If α is an infinite branch of Π and F^t is applicable at some node on α, then F^t is applied at some later node on α.

We construct Π and g as the limit of finite stages, where at stage i we have constructed a finite pre-proof Π_i and a partial function $g_i : S \to \Pi_i$. At every stage we make sure that g_i and Π_i satisfy the following conditions:

7. All open leaves of Π_i are in the range of g_i.
8. All nodes $v \in S$ for which $g_i(v)$ is defined satisfy $\Phi_v = \widetilde{\Sigma}_{g_i(v)}$.

In the base case we define Π_0 to consist of just one node r that is labelled with the sequent Φ^f. The partial function g_0 maps r to v_I. Clearly, this satisfies the conditions 7 and 8.

In the inductive step we consider any open leaf m of Π_i, which has a minimal distance from the root of Π_i. This ensures that in the limit every open leaf is eventually treated, so that Π will not have any open leaves. By condition 7 there is a $u \in S$ such that $g(u) = m$. Our plan is to extend the proof Π_i at the open leaf m to mirror the rule that is applied at u in $\mathbb{T}$. In general this is possible because by condition 8 the formulas in the annotated sequent at $m = g_i(u)$ are the same as the formulas at u. All children of u that are in S should then be mapped by g_{i+1} to new open leaves in Π_{i+1}. Two technical issues feature in all the cases.

First, to ensure that condition 6 is satisfied by our construction we will apply F^t at m, whenever it is applicable. Thus, we need to check whether all formulas in the sequent of m are annotated with u. If this is the case then we apply the total focus rule and proceed with its premise n; otherwise we just proceed with $n = m$. Note that in either case the sequent at n contains the same formulas as the sequent at m and if $n \neq m$ then the trace relation relates the formulas at n in an obvious way to those at m. The second technical issue is that to ensure condition 3 we may need to apply W to the new leaves of Π_{i+1}. For the details of the construction, which are based on a straightforward case distinction depending on the rule $\mathsf{Q}(u)$, we refer to the technical report [18].

We define $\Pi = (T, P, \Sigma, \mathsf{R})$ and the function $g : S \to T$ as the limit of the structures Π_i and the maps g_i, respectively. The proof that g and Π satisfy the conditions 1–6, is fairly routine; details can be found in [18].

It is more interesting to see why Π is a correct Focus_∞-proof. Leaving the routine argument as to why Π is a pre-proof to the reader, we concentrate on the proof that every infinite branch of Π is successful. Let $\beta = (v_n)_{n\in\omega}$ be such a branch. Based on our construction it will not be hard to see that β witnesses infinitely many application of the box rule $\mathsf{R}_\Box$. Our key claim is that

from some moment on, every sequent on β contains a formula in focus. (2)

By condition 2 we can link β to a branch $\alpha = (t_n)_{n\in\omega}$ in S such that there are $0 = k_0 < k_1 < k_2 < \cdots$ with $g(t_i) = v_{k_i}$ for all $i < \omega$. Because α, as a match of the tableau game, is won by Prover, it contains a ν-trail $(\varphi_n)_{n\in\omega}$, so by condition 4 we obtain an annotated trail $\tau = (\psi_n^{a_n})_{n\in\omega}$ on β such that $\varphi_i = \psi_{k_i}$ for all i. Then by Proposition 1 τ is a ν-trail as well; in particular, it contains *no* μ-formulas after a certain moment k.

Now distinguish cases. If β has no application of F^t after k, then by condition 6 this rule is not applicable any more, so that by its definition β must witness a formula in focus at every node v_n with $n \geq k$ indeed. On the other hand, if $\mathsf{R}(v_n) = \mathsf{F}^t$ for $n \geq k$, then at stage $n+1$ every formula is in focus . In particular, we find $a_{n+1} = f$, and since no μ-formula is unfolded on τ after this, we may show that τ keeps passing through formulas in focus from this moment on.

This proves (2), and, again by condition 6, we may conclude that β features only finitely many applications of F^t. Since all applications of F in Π are part of F^t, and the unfocus rule U is not used anywhere in Π, β is successful indeed. □

6 Conclusion & Questions

In this paper we saw that the idea of placing formulas in *focus* can be extended from the setting of logics like LTL and CTL [17] to that of the alternation-free modal μ-calculus: we designed a very simple and natural, cut-free sequent system which is sound and complete for all validities in the language consisting of all (guarded) formulas in the alternation-free fragment $\mathcal{L}_\mu^{af}$ of the modal μ-calculus.

In a follow-up paper we use the Focus system to show that the alternation-free fragment enjoys the Craig Interpolation Theorem. Clearly, these results support the claim that $\mathcal{L}_\mu^{af}$ is an interesting logic with good meta-logical properties.

Below we list questions for future research. To start with, we based our soundness and completeness proofs on Niwiński & Walukiewicz' tableau games [21]. A reviewer suggested that our proofs might be simplified by connecting to the non-wellfounded proof system of Studer [24]. We leave this for future work.

Probably the most obvious question is whether the restriction to guarded formulas can be lifted. Note that guardedness is related to the condition that successful branches in a Focus_∞-proof feature infinitely many applications of the rule $\mathsf{R}_\Box$, which plays a crucial role in the soundness proof (cf. Proposition 4). Without guardedness, this condition would be too strong since it would disqualify any proof for a valid formula like $\nu x.x$.

Note that our proof systems are cut free, and that it follows from our soundness and completeness results that the cut rule is admissible. It would be of interest to see whether this can also be proved constructively, corresponding to a cut elimination procedure for the version of the system with the cut rule.

Another question is whether we may tidy up the focus proof system, in the same way that Afshari & Leigh did with the Jungteerapanich-Stirling system [1, 14, 23]. As a corollary of this it should be possible to obtain an annotation-free sequent system for the alternation-free fragment of the μ-calculus, and to prove completeness of Kozen's axiomatisation for $\mathcal{L}_\mu^{af}$.

It is straightforward to generalise our result to the alternation-free fragment of variants of the modal μ-calculus, such as the polymodal or the monotone μ-calculus. Of particular interest is the *linear time μ-calculus* (i.e., where both $\Diamond$ and $\Box$ are the next time operator), since in this setting the alternation-free μ-calculus is known to have the same expressive power as the full language. It would be interesting to prove a general result for *coalgebraic modal μ-calculi*.

Moving in a somewhat different direction, we are interested to see to which degree the focus system can serve as a basis for sound and complete derivation systems for the alternation-free validities in classes of frames satisfying various kinds of frame conditions.

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