Frame Definability in Conditional Logic

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Abstract

In this paper we investigate classes of finite partially ordered sets that are definable by non-nested formulas in conditional logic. We discuss examples of such definable classes and introduce the notion of a c-morphism between posets as a tool to show that a class of finite posets is not definable. Using an analogue of the Jankov-Fine formulas from modal logic we show that a class of finite posets is definable by a set of formulas if and only if it is closed under c-morphic images. Lastly, we prove a correspondence theorem stating that every class of finite posets that is definable by a formula without nested conditionals is also definable by a first-order formula.

1 Introduction

Conditional logic is a non-normal modal logic that extends propositional logic with a binary modality $\rightsquigarrow$ that is called the (counterfactual) conditional. The guiding semantic intuition is that a conditional $\varphi \rightsquigarrow \psi$ is true if its consequent ψ is true at all worlds that are either most preferred, most plausible or maximally similar to the actual world among all the worlds that make the antecedent φ true. This intuition can be made precise considering an order over the set of worlds and then defining $\varphi \rightsquigarrow \psi$ to be true if ψ is true in all the minimal φ-worlds (Hansson, 1969; Lewis, 1973). The same semantic clause is also used in other settings that are closely related to conditional logic, such as in default reasoning (Kraus et al., 1990; Shoham, 1988) and in belief revision theory (Grove, 1988; Rott, 2009).

The set of validities of conditional logic depends on what class of orders the semantics is based on. Lewis' conditional logic from (Lewis, 1973) consists of all formulas that are valid with the above semantic class over the class of model based on weak orders. A generalization of this logic was obtained later by Burgess (1981) and Veltman (1985), who axiomatize the validities over the class of all partial orders. In the literature on default reasoning and belief revision theory the validities of further classes of posets have been investigated, such as the class of interval orders (Lehmann & Magidor, 1992; Makinson, 1994) or the class of semiorders (Peppas & Williams, 2014; Rott, 2014).

In this paper we systematically investigate the relation between validity in conditional logic and classes of finite posets. To this aim we adapt ideas from frame correspondence theory for normal modal logics (Blackburn et al., 2002, ch. 3) to conditional logic. The central notion for this paper is that of a class of finite posets being definable by a set of formulas in conditional logic. A class of posets is definable by a set of formulas if the class contains precisely the posets over which all the formulas in the set are valid. As an example one has that Lewis' conditional logic defines the class of weak orders.

The main technical contributions of this paper are the following:

1. We provide a formula in conditional logic that defines the class of semitransitive orders. To our knowledge this is a novel result. Characterizations of semitransitive and semiorders, which are semitransitive interval orders, have been given in the context of choice functions (Fishburn, 1975; Jamison & Lau, 1973) and belief revision theory (Peppas & Williams, 2014; Rott, 2014). However, the provided conditions are formulated in the metalanguage and it is not obvious how to express them as axioms in conditional logic.

2. We characterize the definable classes of finite posets as those that are closed under c-morphic images. This results is similar to characterizations in modal logic, which state, roughly, that a class of frames is definable by modal formulas iff it is closed under generated subframes, coproducts and bounded morphic images. As a consequence of our result we get that if a class of finite posets is not definable in conditional logic then there is a concrete counterexample of a c-morphism from a poset that is in the class to a poset that is not in the class.

3. We provide a procedure that, given a formula in conditional logic, computes a first-order formula that is true in exactly those posets where the conditional formula is valid. This result can be seen as a simple version of the Sahlqvist Theorem for conditional logic.

The statement of second result mentioned above makes use of the notion of a c-morphism. This notion is inspired by the notion of a bisimulation between preferential models that was studied in the context of default reasoning by Zhu (2006). A c-morphism in the sense defined in this paper is a function whose graph is a bisimulation in the sense of Zhu (2006). The precise formulation of the conditions in the definition of a c-morphism is quite technical. One part of the definition is the familiar back-condition from the definition of a bounded morphism; however, in general c-morphisms are not order-preserving. The notion of a c-morphism plays a role that is comparable to the notion of a bounded morphism, also called p-morphism, in modal logic. In particular it holds that any two models that are connected by a c-morphism satisfy the same formulas.

To our knowledge this paper is the first study of frame definability in conditional logic. As such our approach still has the following limitations:

1. We only consider formulas of conditional logic in which the conditional is not nested and all occurrences of propositional letters are in the scope of the conditional. This allows us to work with models that are based on a single poset, which means that classes of definable frames will correspond to natural classes of posets. If we were to work with formulas that contain nested conditionals we would need to consider frames that are comparative similarity relations, see (Lewis, 1973, sec. 2.3) or (Burgess, 1981; Veltman, 1985), which are families of world-indexed posets.

2. We only consider finite posets. The main reason for this is that over infinite orders minimization does not yield a well-behaved conditional logic. This is a well-known problem with the order semantics of conditional logic. Two solutions have been proposed in the literature. First, one can make the so-called limit assumption, which is to require infinite posets to be well-founded (Lewis, 1973, sec. 1.7). Over well-founded posets the usual axioms of conditional logic remain valid, however, the logic looses the compactness property. Second, one can change the semantic clause of the conditional to a more complicated clause under which all of the usual axioms are also validated over well-founded orders (Burgess, 1981). However, in Appendix A we provide an example which suggests that with this semantic clause frame definability is not well-behaved.

3. We only consider frame definability relative to posets. It is possible to give a semantics of the conditional over preorders. However, both Burgess (1981) and Veltman (1985) show that the logic of the class of preorders is the same as the logic of the more restricted class of posets. It is in fact quite easy to see that for every model over a preorder we can define a model over a poset that satisfies the same conditionals. The idea is to split up each cluster of equicomparable worlds into an anti-chain of mutually incomparable worlds, while keeping all other comparisons the same.

The structure of this paper is as follows: In Section 2 we discuss discuss the syntax and semantics of conditional logic and the notions of validity and definable classes. In Section 3 we provide examples of definable classes of posets that have arisen in the literature. In Section 4 we introduce the notion of a c-morphism, which we then use in Section 5 to prove for some examples of classes of posets that they are not definable. Section 6 contains the proof of the characterization result that a class of finite posets is definable iff it is closed under c-morphic images. In Section 7 we discuss how to compute a first-order correspondent for every non-nested formula of conditional logic.

2 Preliminaries

In this section we discuss the language of conditional logic and the semantics of formulas in this language over posets. We also define the notion of a definable class of posets.

2.1 Syntax

Conditional logics are commonly formulated in a classical propositional modal language with one binary modality $\rightsquigarrow$. A formula of the form $\varphi \rightsquigarrow \psi$ is called a *conditional* with *antecedent* φ and *consequent* ψ. That $\rightsquigarrow$ is a modality means that one can nest conditionals, as for example in the formula $(((p \rightsquigarrow q) \rightsquigarrow r) \wedge q) \to r$. In this paper we only consider formulas in which the conditional is not nested and all propositional letters are in the scope of a conditional. To make this precise fix an infinite set Prop of propositional letters and consider the grammar:

$$\begin{aligned} \varphi_0 &::= p \mid \neg\varphi_0 \mid \varphi_0 \wedge \varphi_0, && \text{where } p \in \mathsf{Prop}, \\ \varphi_1 &::= \varphi_0 \rightsquigarrow \varphi_0 \mid \neg\varphi_1 \mid \varphi_1 \wedge \varphi_1. \end{aligned}$$

Let $\mathcal{L}_0$ be the set of formulas generated from φ_0 and $\mathcal{L}_1$ the set of formulas generated from φ_1. Note that $\mathcal{L}_0$ is just the language of classical propositional logic. In both $\mathcal{L}_0$ and $\mathcal{L}_1$ we use further Boolean connectives, such as $\vee$, $\to$, and $\leftrightarrow$, as abbreviations with their usual meaning in classical logic. To omit parenthesis we assume that $\neg$ binds stronger than $\wedge$ and $\vee$, which in turn bind stronger than $\rightsquigarrow$, $\to$ and $\leftrightarrow$.

We are going to focus on formulas from $\mathcal{L}_1$ that are of the shape

$$\bigwedge_{i=1}^{n} (\varphi_i \rightsquigarrow \psi_i) \to \bigvee_{j=1}^{m} (\gamma_j \rightsquigarrow \delta_j),$$

where $\varphi_i, \psi_i, \gamma_j, \delta_j \in \mathcal{L}_0$ for all i and j. We call such formulas *inference rules* or simply *inferences* or *rules* and suggestively write them as

$$\frac{\varphi_1 \rightsquigarrow \psi_1 \quad \dots \quad \varphi_n \rightsquigarrow \psi_n}{\gamma_1 \rightsquigarrow \delta_1 \quad \dots \quad \gamma_m \rightsquigarrow \delta_m},$$

or as Σ/Γ, where $\Sigma = \{\varphi_i \rightsquigarrow \psi_i \mid 1 \leq i \leq n\}$ and $\Gamma = \{\gamma_j \rightsquigarrow \delta_j \mid 1 \leq j \leq m\}$. The elements of Σ are also called the *premises* of the inference Σ/Γ and the elements of Γ are its *conclusions.* We allow for the cases where Σ is empty, meaning that the inference corresponds to a formula of the form $\top \to \bigvee \Gamma$, and where Γ is empty, meaning that the inference corresponds to the formula $\bigwedge \Sigma \to \bot$.

It is a consequence of Corollary 1 below that for the purpose of understanding classes of posets that are definable by a formula in $\mathcal{L}_1$ it suffices to only consider formulas that are in the shape of inference rules. Focusing on the presentation of formulas as inference rules also matches the presentation in the setting of non-monotonic consequence relations, where such rules between conditional, thought of as non-monotonic inference relations, are taken as basic (Kraus et al., 1990). In Section 3 we provide multiple natural examples of such inference rules that have been discussed in the literature.

2.2 Semantics

The semantics of the conditional in conditional logic can be given in terms of ternary similarity relations $\leq$ where $u \leq_w v$ holds if u is at least as similar to w as v (Lewis, 1973, sec. 2.3). A conditional then holds with respect to such a relation if the consequent is true at all the worlds that satisfy the antecedent and are maximally similar to the actual world among the wolds that satisfy the antecedent. In our setting, where we do not allow for the nesting of conditionals, one can however omit the relativization to the actual world and just minimize relative to a poset. This leads to the notion of a preferential model, which is the kind of semantic structure that is commonly used for non-monotonic consequence relations (Kraus et al., 1990):

Definition 2.1 (Preferential Model)**.** Recall that a poset $P = (W, \leq)$ is a set W together with a partial order $\leq$ on W, where a partial order is just a reflexive, transitive and anti-symmetric relation $\leq \subseteq W \times W$. A *preferential model* $M = (W, \leq, V)$ is a poset $(W, \leq)$ together with a function $V : \mathsf{Prop} \to \mathcal{P}W$. The elements of W are called *worlds* and the function V is called the *valuation function*.

In this paper we assume that, unless explicitly specified otherwise, all posets and models are finite, meaning that the set of worlds W is a finite set.

The semantics of formulas from $\mathcal{L}_0$ and $\mathcal{L}_1$ is defined in the standard way. Thus, the set of worlds $[\![\varphi]\!]_V \subseteq W$ where a formula $\varphi \in \mathcal{L}_0$ is true is computed by recursion on the complexity of the formula using the valuation function V in the base case:

$$[\![p]\!]_V = V(p), \qquad [\![\neg\varphi]\!]_V = W \setminus [\![\varphi]\!]_V, \text{ and } \qquad [\![\varphi \wedge \psi]\!]_V = [\![\varphi]\!]_V \cap [\![\psi]\!]_V.$$

We often write $[\![\varphi]\!]$ for $[\![\varphi]\!]_V$ if V is clear from the context.

We use the standard clauses for the propositional connectives over $\mathcal{L}_1$ relative to the model $M = (W, \leq, V)$:

$$M \models \neg\varphi \text{ iff not } M \models \varphi, \text{ and } \qquad M \models \varphi \wedge \psi \text{ iff } M \models \varphi \text{ and } M \models \psi.$$

For the semantics of the conditional we use the order $\leq$. A conditional is true if all of the minimal antecedent worlds satisfy the consequent:

$$M \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad \mathrm{Min}(\leq)([\![\varphi]\!]) \subseteq [\![\psi]\!].$$

The minimal worlds of a set $A \subseteq W$ in a partial order $\leq$ over W can be defined as $\mathrm{Min}(\leq)(A) = \{x \in A \mid \forall y \in A(y \leq x \implies x \leq y)\}$.

The following alternative formulation of the semantic clause is useful for some results in this paper:

Proposition 1. Relative to all finite models $(W, \leq, V)$ it holds that

$$M \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad \text{for all } x \in [\![\varphi]\!] \text{ there is a } y \leq x \text{ with } y \in [\![\varphi]\!] \cap [\![\psi]\!].$$

Proof. For the left-to-right direction, assume that $M \models \varphi \rightsquigarrow \psi$. Then, by the semantics of the conditional we get that $\mathrm{Min}(\leq)([\![\varphi]\!]) \subseteq [\![\psi]\!]$. The right hand side follows if we can show that for every $x \in [\![\varphi]\!]$ there is a $y \in \mathrm{Min}(\leq)([\![\varphi]\!])$ with $y \leq x$. We derive a contradiction from the assumption that this is not the case, meaning that there is some $x \in [\![\varphi]\!]$ such that $y \notin \mathrm{Min}(\leq)([\![\varphi]\!])$ for all $y \leq x$. Using this assumption we can inductively define a infinite descending chain $x_0 > x_1 > \ldots$ such that $x_i \in [\![\varphi]\!]$ and $x_i \leq x$ for all i. This constradicts that assumption of the theorem that the model is finite. In the base case of the induction we take $x_0 = x$. For the inductive step assume that we have a $x_i \in [\![\varphi]\!]$ with $x_i \leq x$. From $x_i \leq x$ it follows that $x_i \notin \mathrm{Min}(\leq)([\![\varphi]\!])$. Thus there is some $x' \in [\![\varphi]\!]$ such that $x' < x_i$. By transitivity we get that $x' \leq x$ and hence we can set $x_{i+1} = x'$.

For the other direction, assume that for every $x \in [\![\varphi]\!]$ there is $y \leq x$ with $y \in [\![\varphi]\!] \cap [\![\psi]\!]$. Let $x \in \mathrm{Min}(\leq)([\![\varphi]\!])$ be arbitrary. By assumption, there must be some $y \leq x$ with $y \in [\![\varphi]\!] \cap [\![\psi]\!]$. As x is minimal in $[\![\varphi]\!]$ wrt. $\leq$ it follows that $y = x$, i.e. $x \in [\![\psi]\!]$. Therefore, we get that $M \models \varphi \rightsquigarrow \psi$. □

The notions of validity and of a definable class are then defined analogously to how they are defined for normal modal logics:

Definition 2.2 (Validity)**.** A formula $\varphi \in \mathcal{L}_1$ is *valid* in a poset $P = (W, \leq)$ if for all valuations $V : \mathsf{Prop} \to \mathcal{P}W$ we have that $M \models \varphi$ for the model $M = (W, \leq, V)$. We can extend validity to sets of formulas and classes of posets: A set of formulas $\Sigma \subseteq \mathcal{L}_1$ is valid in a class of posets $\mathcal{C}$ if φ is valid in P for all formulas $\varphi \in \Sigma$ and P in the class $\mathcal{C}$. If a formula, or set of formulas, is not valid in a poset, or class of posets, we also say that the formula, or set of formulas, is *falsifiable* in the poset, or class of posets and that the poset, or class of posets, *falsifies* the formula, or set of formulas.

Definition 2.3 (Definable class)**.** A formula $\varphi \in \mathcal{L}_1$ *defines* a class of finite posets $\mathcal{C}$ iff for all finite posets P it holds that

$$P \text{ is in } \mathcal{C} \quad \text{iff} \quad \varphi \text{ is valid in } P.$$

Similarly, a set of formulas $\Sigma \subseteq \mathcal{L}_1$ *defines* a class of finite posets $\mathcal{C}$ iff for all finite posets P it holds that P is in $\mathcal{C}$ iff Σ is valid in P. A class of finite posets $\mathcal{C}$ is *definable* if there is some set $\Sigma \subseteq \mathcal{L}_1$ such that Σ defines $\mathcal{C}$.

We conclude this section by showing that in order to study the definable classes it suffices to consider formulas that are inference rules in the sense defined above.

Proposition 2. Every formula $\varphi \in \mathcal{L}_1$ is equivalent to a conjunction of inference rules.

Proof. This follows with propositional reasoning. To this aim consider $\varphi \in \mathcal{L}_1$ as a propositional formula where the conditionals are atoms. It is clear that we can rewrite φ into an eqivalent conjunctive normal form, i.e. a conjunction of disjunctions of literals. It is clear that every such disjunction of literals

$$\bigvee_{i=1}^{n} \neg(\varphi_i \rightsquigarrow \psi_i) \vee \bigvee_{j=1}^{m} (\gamma_j \rightsquigarrow \delta_j)$$

is propositionally equivalent to the inference

$$\bigwedge_{i=1}^{n} (\varphi_i \rightsquigarrow \psi_i) \to \bigvee_{j=1}^{m} (\gamma_j \rightsquigarrow \delta_j).$$

□

Corollary 1. For every formula $\varphi \in \mathcal{L}_1$ there is a finite set of inferences $\sigma = \{\Sigma_1/\Gamma_1, \dots, \Sigma_n/\Gamma_1\}$ such that σ is valid in a poset $(W, \leq)$ iff φ is valid in $(W, \leq)$.

Proof. By Proposition 2 we can write every formula $\varphi \in \mathcal{L}_1$ as a conjunction of inference rules $\Sigma_1/\Gamma_1, \dots, \Sigma_n/\Gamma_n$. Moreover, a conjunction is valid in a poset P iff each of its conjuncts is valid in P. Thus φ is valid in P if all Σ_i/Γ_i for $i \in \{1, \dots, n\}$ are valid in P. □

3 Definable classes of posets

In this section we discuss examples of classes of finite posets that are definable by a formula in conditional logic. Figure 1 provides an overview of the examples from this section. The first-order formulas that describe the classes in the second column should be understood such that all free variables are universally quantified. The rules defining linear orders and orders with a minimum have an empty set of premises.

Example 1. (Antichains) The class of antichains is defined by the rule

$$\frac{\top \rightsquigarrow p}{\neg p \rightsquigarrow \bot.}$$

class	first-order formula	defining rule
antichain	$x \leq y \to y \leq x$	$\dfrac{\top \rightsquigarrow p}{\neg p \rightsquigarrow \bot}$
has a minimum	$\exists x \forall y (x \leq y)$	$\dfrac{}{\top \rightsquigarrow p \quad \top \rightsquigarrow \neg p}$
linear order	$x \leq y \vee y \leq x$	$\dfrac{}{p \rightsquigarrow q \quad p \rightsquigarrow \neg q}$
weak order	$x < y \to z < y \vee x < z$	$\dfrac{p \rightsquigarrow q}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}$
interval order	$x < y \wedge z < u \to x < u \vee z < y$	$\dfrac{p \vee r \rightsquigarrow q}{p \rightsquigarrow q \quad r \rightsquigarrow q}$
semitransitive	$x < y \wedge y < z \to x < u \vee u < z$	$\dfrac{p \rightsquigarrow q \quad p \wedge \neg q \rightsquigarrow \neg r}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}$

Figure 1: Examples of definable classes

In order to see this, suppose that $P = (W, \leq)$ is a poset which is not an antichain, i.e. there are two points x and y such that $x < y$. Then, under the valuation $V(p) = \{x\}$, it is true that $\top \rightsquigarrow p$ but $\mathrm{Min}(\leq)([\![\neg p]\!]) = \{y\} \not\subseteq [\![\bot]\!] = \emptyset$.

Viceversa, if an antichain $P = (W, \leq)$ satisfies $\top \rightsquigarrow p$ it means that all the minimal elements of P satisfy p. But since P is an antichain, all its elements are minimal and because $[\![p]\!] \cap [\![\neg p]\!] = \emptyset$ it must be $[\![\neg p]\!] = \emptyset$, therefore $\emptyset \subseteq [\![\bot]\!]$. In Appendix B.1 we discuss how this example can be generalized to show that for every number n the class of all posets with chains of length at most n is definable.

Example 2 (Orders with a minimum)**.** It is easy to check that

$$\frac{}{\top \rightsquigarrow p \quad \top \rightsquigarrow \neg p}$$

defines the class of all posets that have a unique minimal element. We show in Appendix B.2 how the rule can be generalized to characterize the classes of posets with at most n minimal elements.

Example 3 (Linear orders)**.** We leave it to the reader to convince themself that the rule

$$\frac{}{p \rightsquigarrow q \quad p \rightsquigarrow \neg q}$$

defines the class of all linear orders. Note that this rule has no premises and thus is equivalent to the formula $(p \rightsquigarrow q) \vee (p \rightsquigarrow \neg q)$. Lewis (1973, sec. 3.4) calls the law expressed by this formula "conditional excluded middle" and considers its validity to be the characteristic of Stalnaker's account of conditionals from (Stalnaker, 1968).

Example 4 (Weak orders)**.** The class of weak orders is defined by the rule

$$\frac{p \rightsquigarrow q}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}.$$

Note that weak orders are also called strict weak orders and they can also be represented as total preorders. They provide a semantics for conditional logic that is equivalent to Lewis' systems of spheres (Lewis, 1973, sec. 2.3). In belief revision theory they provide a semantics for the classic AGM postulates for belief revision (Alchourrón et al., 1985; Grove, 1988). In the guise of total preorders they are also behind the standard semantics of dynamic epistemic logic (Baltag & Smets, 2006). The inference rule used here to define weak order is just the rule of “rational monotonicity” from (Lehmann & Magidor, 1992).

Example 5 (Interval orders)**.** It is not hard to show that the following rule defines interval orders

$$\frac{p \vee r \rightsquigarrow q}{p \rightsquigarrow q \quad r \rightsquigarrow q}.$$

This rule has been called “disjunctive rationality” in the literature on non-monotonic reasoning (Lehmann & Magidor, 1992; Makinson, 1994). Interval orders derive their name from the observation that they are precisely the orders that can be represented by a natural ordering of arbitrary length intervals on the line (Fishburn, 1970). The interval order condition also plays a role in the semiorders that we discuss as the next example.

Example 6 (Semitransitive orders and semiorders)**.** Semitransitive orders are defined by the inference rule

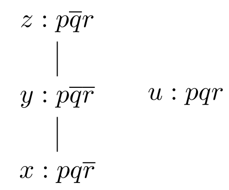
$$\frac{p \rightsquigarrow q \quad p \wedge \neg q \rightsquigarrow \neg r}{p \rightsquigarrow \neg r \quad p \wedge r \rightsquigarrow q}\ . \tag{1}$$

Note first that semitransitivity generalizes the condition for weak orders. In Appendix B.3 we discuss a further generalization of this notion to n-weak orders for every number n and provide the defining inference rules.

As semiorders are semitransitive interval orders, it follows that semiorders are defined by this rule together with the disjunctive rationality rule from Example 5. It has been argued that semiorders model human preference more adequately than weak orders (Luce, 1956; van Rooij, 2011). It can be shown that they are representable, analogously to interval orders, by intervals of constant length (Scott & Suppes, 1958). Semiorders have been axiomatized in the context of choice functions (Fishburn, 1975; Jamison & Lau, 1973) and in belief revision theory (Peppas & Williams, 2014; Rott, 2014). However, both settings use axioms that are not obviously expressible in the language of conditional logic.

We now prove that the rule in (1) above defines the class of semitransitive posets.

First assume that $P = (W, \leq)$ is a poset that fails to be semitransitive. This means that there are points x, y, z and u with $x < y < z$ such that u is incomparable to x and z. Thus, P has a subposet that looks as follows:



It is easy to see that in the model with a valuation V such that $V(p) = \{x, y, z, u\}$, $V(q) = \{x, u\}$ and $V(r) = \{z, u\}$ all the premises of the rule in (1) are true while all the conclusions are false.

For the other direction assume that the rule in (1) is valid in a poset $P = (W, \leq)$. To see that then P is semitransitive choose any points x, y, z and u in P such that $x < y < z$. To show that then either $x < u$ or $u < z$ consider a valuation V with $V(p) = \{x, y, z, u\}$, $V(q) = \{x, u\}$ and $V(r) = \{z, u\}$. It is clear that this makes the conditionals $p \rightsquigarrow q$ and $p \wedge \neg q \rightsquigarrow \neg r$ true because $\mathrm{Min}(\leq)([\![p]\!]) \subseteq \{x, u\} = [\![q]\!]$ and $\mathrm{Min}(\leq)([\![p \wedge \neg q]\!]) = \mathrm{Min}(\leq)(\{y, z\}) = \{y\} \subseteq [\![\neg r]\!]$. It follows that either $p \rightsquigarrow \neg r$ or $p \wedge r \rightsquigarrow q$ is true in $(W, \leq, V)$. The former means that $\mathrm{Min}(\leq)(\{x, y, z, u\}) \subseteq \{x, y\}$, which entails $x < u$, and the latter means that $\mathrm{Min}(\leq)(\{z, u\}) \subseteq \{x, u\}$, which means that $u < z$.

4 C-morphisms

In this section we introduce c-morphisms, which are our main tool to show that a class of posets is not definable by a formula in conditional logic. Our notion of a c-morphism is an adaptation of the the notion of a bisimulation between preferential models by Zhu (2006). C-morphisms are precisely those functions, whose graph is a bisimulation in the sense of Zhu (2006).

Definition 4.1 (C-morphism)**.** A *c-morphism* f from a poset $P = (W, \leq)$ to a poset $P' = (W', \leq')$ is a function $f : W \to W'$ such that:

(i) For all $w \in W$ and $u' \leq' f(w)$ there is a $u \leq w$ such that $f(u) = u'$.

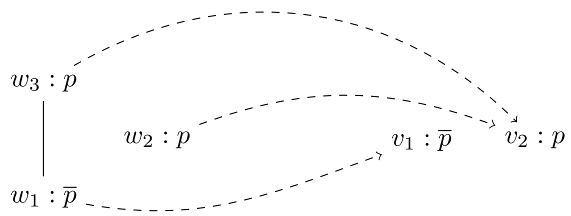
(ii) For all $w' \in W'$ there is a $w \in W$ such that $f(w) = w'$ and for all $u \leq w$ we have that $f(u) \leq' w'$.

We can extend the notion of a c-morphism to models such that $f : W \to W'$ is a c-morphism from $M = (W, \leq, V)$ to $M' = (W', \leq', V')$ if f is a c-morphism from $(W, \leq)$ to $(W', \leq')$ and $V(p) = f^{-1}(V'(p))$ for all $p \in \mathsf{Prop}$.

We call a poset P' a *c-morphic image* of a poset P if there is some c-morphism from P to P'.

Note that the first condition in the definition of c-morphisms is just the back condition for bounded morphisms in modal logic. Also note that it follows from the second condition that every c-morphism is surjective. However, c-morphisms need not be order-preserving, as the following example shows:

Example 7. The following is an picture of a c-morphism between two preferential models. The first models contains the worlds w_1, w_2 and w_3 and the second model contains v_1 and v_2. The mapping of the c-morphism is depicted with the dashed arrows.



C-morphisms can be characterized in terms of the preservation of minimal elements in the following sense:

Proposition 3. Let $P = (W, \leq)$ and $P' = (W', \leq')$ be finite posets, $f : W \to W'$ is a c-morphism if and only if for every subset $X' \subseteq W'$ we have that $f(\mathrm{Min}(\leq)(f^{-1}(X'))) = \mathrm{Min}(\leq')(X')$.

Proof. For the left-to-right one, assume that $f : W \to W'$ is a conditional morphism from P to P'. We prove that, for any $X' \subseteq W'$, $f(\mathrm{Min}(\leq)(f^{-1}(X'))) = \mathrm{Min}(\leq')(X')$ by showing the double inclusion of the two sets:

- $f(\mathrm{Min}(\leq)(f^{-1}(X'))) \subseteq \mathrm{Min}(\leq')(X')$: let $w' \in f(\mathrm{Min}(\leq)(f^{-1}(X')))$. Hence there must be $w \in \mathrm{Min}(\leq)(f^{-1}(X'))$ such that $f(w) = w'$. To prove that $w' \in \mathrm{Min}(\leq')(X')$ it suffices to show that, for any $u' \in X'$ with $u' \leq' w'$, $u' = w'$. Thus, let u' be any such world. By condition (i) from the definition of c-morphisms we have that there is $u \leq w$ with $f(u) = u'$. Hence, $u \in f^{-1}(X')$. As $w \in \mathrm{Min}(\leq)(f^{-1}(X'))$ we have that $u = w$, hence $u' = w'$, proving the claim.

- $f(\mathrm{Min}(\leq)(f^{-1}(X'))) \supseteq \mathrm{Min}(\leq')(X')$: let $w' \in \mathrm{Min}(\leq')(X')$. By condition (ii) for conditional morphisms there is some $w \in W$ such that $f(w) = w'$ and for all $u \leq w$ we have that $f(u) \leq' w'$. We will show that there is some world $s \in \mathrm{Min}(\leq)(f^{-1}(X'))$ with

$f(s) = w'$, proving the claim as this implies that $w' \in f(\mathrm{Min}(\leq)(f^{-1}(X')))$. Let $u \leq w$ with $u \in f^{-1}(X')$: then $f(u) = u' \in X'$. Moreover, we also get that $u' \leq' w'$, from which it follows that $u' = w'$ as $w' \in \mathrm{Min}(\leq')(X')$. Hence, either $u = w$ or not: in the former, the claim is shown, as it means that $w \in \mathrm{Min}(\leq)(f^{-1}(X'))$. In the latter we also have two cases: either $u \in \mathrm{Min}(\leq)(f^{-1}(X'))$ or not. If u is minimal then the claim follows again. Otherwise, there must be some distinct $v \leq u$ with $v \in f^{-1}(X')$, i.e. $f(v) = v' \in X'$. By transitivity of $\leq$ it follows that $v \leq w$, hence $v' \leq' w'$, which implies that $v' = w'$ as $w' \in \mathrm{Min}(\leq')(X')$. Therefore, this same argument can always be repeated thanks to the transitivity of $\leq$, however due to the finiteness of W it follows that there must be some $s \in \mathrm{Min}(\leq)(f^{-1}(X'))$ with $f(s) = w'$, which proves the claim.

For the right-to-left direction, assume that f is such that for any $X' \subseteq W'$ we have that $f(\mathrm{Min}(\leq)(f^{-1}(X'))) = \mathrm{Min}(\leq')(X')$. We show that f satisfies both conditions for conditional morphisms:

(i) Let $w \in W$ and $u' \leq' f(w) = w'$ be arbitrary. We show that for some $u \leq w$ we have that $f(u) = u'$. If $u' = w'$ then by reflexivity it trivially follows that that $w \leq w$ and $f(w) = u'$. Hence, assume that $u' \neq w'$. Take $X' = \{u', w'\}$: then by assumption we get that $f(\mathrm{Min}(\leq)(f^{-1}(X'))) = \mathrm{Min}(\leq')(X') = \{u'\}$. Hence, as $w' \notin \mathrm{Min}(\leq')(X')$ we have that $w \notin \mathrm{Min}(\leq)(f^{-1}(X'))$. Therefore, we deduce that there must be some $u \leq w$ with $u \in \mathrm{Min}(\leq)(f^{-1}(X'))$. As u is minimal in $f^{-1}(X')$, it follows that $f(u) = u'$, proving the claim;

(ii) Let $w' \in W'$ be arbitrary and assume by way of contradiction that there is no $w \in W$ such that $f(w) = w'$ and for any $u \leq w$ we have that $f(u) \leq' w'$. Obviously it can't be that there is no $w \in W$ such that $f(w) = w'$ (take $X' = \{w'\}$). Hence, all $w \in W$ are such that, for some $u \leq w$, we have that $f(u) = u'$ and it is not the case that $u' \leq' w'$, which we will denote with $u' \not\leq' w'$ from now on. Observe that from this it trivially follows that $u' \neq w'$, hence also $u \neq w$. Then, take $X' = \{u', w'\}$. By assumption we know that $w' \in \mathrm{Min}(\leq')(X')$, however $w \notin \mathrm{Min}(\leq)(f^{-1}(X'))$ as $\{u, w\} \subseteq f^{-1}(X')$ and u is strictly below w. Then, there must be some $v \in W$ such that $f(v) = w'$ and $v \in \mathrm{Min}(\leq)(f^{-1}(X'))$, i.e. there is no $s \leq v$ such that $f(s) = w'$ or $f(s) = u'$. Again, by assumption, there must be $s \leq v$ such that $f(s) = s' \not\leq' w'$. Hence, we can repeat the same argument as before by taking $X'' = \{s', u', w'\}$. Now, observe that no matter how many worlds from W' we add, we'll always have that w' is minimal in these sets. Hence, we can repeat this process an arbitrary number of times, obtaining an infinite chain of strictly increasing sets $X' \subsetneq X'' \subsetneq \cdots \subsetneq X^n \subsetneq \ldots$, which imply that W' is an infinite set, contradicting the fact that P' is a finite poset.

□

The crucial property of c-morphisms is that they preserve the truth of formulas in conditional logic:

Proposition 4. Let $M = (W, \leq, V)$ and $M' = (W', \leq', V')$ be finite preferential models and f a c-morphism from M to M'. Then, it holds for all formulas $\varphi \in \mathcal{L}_1$ that

$$M \models \varphi \quad \text{iff} \quad M' \models \varphi.$$

Proof. The proof is an induction on the complexity of formulas $\varphi \in \mathcal{L}_1$.

First note that it follows with an easy induction on the complexity of formulas $\varphi \in \mathcal{L}_0$ that $[\![\varphi]\!]_V = f^{-1}([\![\varphi]\!]_{V'})$. The base case that $V(p) = f^{-1}(V'(p))$ for all $p \in \mathsf{Prop}$ holds because f is a morphism of models.

The main inductive case is where $\varphi = \chi \rightsquigarrow \psi$ is a conditional with $\chi, \psi \in \mathcal{L}_0$. First, observe that because the direct image map is left adjoint to the inverse image map we have that

$$f(\mathrm{Min}(\leq)([\![\varphi]\!]_V)) \subseteq [\![\psi]\!]_{V'} \quad \text{iff} \quad \mathrm{Min}(\leq)([\![\varphi]\!]_V) \subseteq f^{-1}([\![\psi]\!]_{V'}). \tag{2}$$

Because from the base case for formulas in $\mathcal{L}_0$ we have that $f^{-1}(\llbracket\psi\rrbracket_{V'}) = \llbracket\psi\rrbracket_V$ it follows that the right side of (2) is equivalent to $\mathrm{Min}(\leq)(\llbracket\varphi\rrbracket_V) \subseteq \llbracket\psi\rrbracket_V$, which means that $M \models \varphi \rightsquigarrow \psi$. On the other hand using that f is a c-morphism we get from Proposition 3 that $f(\mathrm{Min}(\leq)(f^{-1}(\llbracket\varphi\rrbracket_{V'}))) = \mathrm{Min}(\leq')(\llbracket\varphi\rrbracket_{V'})$. Combining with the $f^{-1}(\llbracket\psi\rrbracket_{V'}) = \llbracket\psi\rrbracket_V$ that we get from the base case it follows that the left side of (2) is equivalent to $\mathrm{Min}(\leq')(\llbracket\varphi\rrbracket_{V'}) \subseteq \llbracket\psi\rrbracket_{V'}$. This is precisely the semantics of $M' \models \varphi \rightsquigarrow \psi$.

The remaining cases for the propositional connectives in $\mathcal{L}_1$ follow with by standard reasoning. □

Using the preservation result from the previous section it is relatively easy to show the following:

Theorem 1. *Let f be a c-morphism from a finite poset $P = (W, \leq)$ to a finite poset $P' = (W', \leq')$. Then every formula $\varphi \in \mathcal{L}_1$ that is valid in P is also valid in P'.*

Proof. We reason by contraposition: assume that φ is not valid in P'. Then, there is some valuation $V' : \mathsf{Prop} \to \mathcal{P}W'$ s.t. for the derived model $M' = (W', \leq', V')$ we have that $M \not\models \varphi$. We lift V' to a valuation $V : \mathsf{Prop} \to \mathcal{P}W$ for P exploiting f in the following way: for $p \in \mathsf{Prop}$ we take $V(p) = f^{-1}(V'(p))$. Let $M = (W, \leq, V)$ be the derived model; by definition we now have that f is a c-morphism between the models M and M'. Hence, by Proposition 3, it follows that $M \not\models \varphi$, i.e. φ is not valid in P. □

One might rephrase Theorem 1 as stating that definable classes of posets are closed under c-morphic images. We can exploit this to show that some class of posets $\mathcal{C}$ is not definable by a formula, or set of formulas, in conditional logic: If we find two posets P and P' such that P is in $\mathcal{C}$ but P' is not in $\mathcal{C}$ and there is a c-morphism from P to P' then $\mathcal{C}$ is not definable.

5 Classes of posets that are not definable

In this section we use Theorem 1 to give examples of classes of posets that are not definable by formulas in conditional logic.

We first make an observation about c-morphic images, which allows us to easily show for many classes of posets that they are not definable. The observation is that for every poset P with n minimal elements there is a c-morphism from P to the antichain with n elements A_n. Therefore, a class of posets not closed under such antichains cannot be definable.

Lemma 1. *Let A_n be the antichain with n elements. Then we have that for every finite poset P with at least n minimal elements there is a c-morphism f from P to A_n.*

Proof. Let $P = (W, \leq)$ be a finite poset with an enumeration $m_1, \ldots, m_k$ of its k distinct minimal elements for $k \geq n$. Let $a_1, \ldots, a_n$ be a enumeration of the elements in the n-element antichain A_n. Then, first define a surjection g from $\{m_1, \ldots, m_k\}$ to $\{a_1, \ldots, a_n\}$. For instance this can be done by setting $g(m_i) = a_i$ for $i \leq n$ and $g(m_i) = a_0$ for $i > n$.

To define the c-morphism f first observe that because P is finite we have that for every $w \in W$ there is some minimal element m_w of P such that $m_w \leq w$. We then define f such that it maps a $w \in W$ to the element $g(m_w)$ of A_n.

The definition of f satisfies condition (i) of Definition 4.1 because for every $w \in W$ and $u' \leq f(w)$ we have that $f(w) = u'$, as A_n is an antichain.

For condition (ii) note that for every $w' \in W$ we have $w' = f(m)$ for some minimal element m of P. If we then consider any $u \leq m$ we get that $u = m$, by the minimality of m, and hence it trivially holds that $f(u) = w'$. □

Together with Theorem 1 we obtain the following corollary:

Corollary 2. *A class of finite posets is not definable if it contains a poset P with k minimal elements but does not contain the antichain A_n with $n \leq k$ elements.*

From this corollary we immediately obtain the following examples of classes that are not definable by formulas in conditional logic:

Example 8. The following classes of finite posets are not definable:

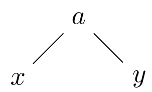
1. posets with more than n elements for $n \geq 1$;
2. posets with exactly n elements for $n > 1$;
3. posets with more than n minimal/maximal elements for $n \geq 1$;
4. posets with chains longer than n for $n \geq 1$.

In each of these classes we have some poset with at least 1 minimal element but all of these classes do not contain the antichain A_1 with exactly one element.

Example 9. The following classes of finite posets are not definable:

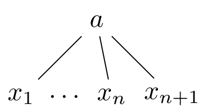
1. connected posets;
2. posets with a maximum;
3. join-semilattices.

Observe that all these classes contain the following poset:



with exactly 2 minimal and 1 maximal element. But all of these classes do not contain the antichain A_2 with 2 elements.

Example 10. As a generalization of the previous example we also get that the class of posets with at most n maximal elements for $n \geq 1$ is not definable. It contains the poset



with $n+1$ minimal and 1 maximal elements. But it does not contain A_{n+1}, the antichain with $n+1$ elements.

Lastly, we have one example of undefinable class for which we could not use the above corollary to prove its undefinability:

Example 11. The class of finite meet-semilattices is not definable. Consider the following two posets:

Clearly, the left poset is a meet-semilattice, but the right one is not. We leave it to the reader to verify that the function f with $f(x) = 4'$ and $f(n) = n'$ for $n \in \{1, 2, \ldots, 6\}$ defines a c-morphism from the left poset to the right one. To check condition(ii) of Definition 4.1 observe that for the element $4'$ on the right we have the element 4 on the left such that for each $n \leq 4$ it holds that $n' \leq 4'$.

6 The characterization theorem

We have already seen in Section 4 that every definable class of finite posets is closed under c-morphic images. In this section we show that the converse is also true: Every class of finite posets that is closed under c-morphic images is definable by a set of formula in conditional logic. Thus, the definable classes are precisely those classes of frames that are closed under c-morphic images:

Theorem 2. A class of finite posets $\mathcal{C}$ is definable by some set of formulas $\Sigma \subseteq \mathcal{L}_1$ if and only if $\mathcal{C}$ is closed under c-morphic images.

Before we prove this theorem let us consider the special case of finite linear orders for which one can give a relatively simple argument:

Remark 1. For finite linear orders one can give a direct proof of Theorem 2. The crucial observation is that a class of finite linear orders $\mathcal{C}$ is closed under c-morphic images if and only if whenever $\mathcal{C}$ contains the linear order with m elements then it also contains all linear orders with n elements for $n \leq m$. Thus, if $\mathcal{C}$ is closed under c-morphic images it is either the class of all linear orders or it is the class of all linear orders of length at most n for some $n \in \omega$. Clearly, the class of all finite linear orders is definiable and we show in Appendix B.1 that the class of posets with chains of length at most n is definable.

To prove Theorem 2 in the general case we define an analogue of the Jankov-Fine formulas that can be used to show a similar characterization result for modal logic over finite transitive frames (Blackburn et al., 2002, sec. 3.4). Thus, we define for every finite poset P a characteristic formula χ_P with the property that for every finite poset P' we get that P' falsifies χ_P iff P is a c-morphic image of P'.

Definition 6.1. Fix a finite poset $P = (W, \leq)$ such that $W = \{w_1, \ldots, w_n\}$. For every point $w_i \in W$ define the formula $\alpha_i \in \mathcal{L}_0$ such that

$$\alpha_i = \bigvee \{p_j \mid w_j \not\leq w_i\}.$$

We then define the *characteristic formula* $\chi_P \in \mathcal{L}_1$ of P as the rule

$$\frac{\neg \bigvee_{j=1}^{n} p_j \rightsquigarrow \bot \quad \{p_i \wedge p_j \rightsquigarrow \bot \mid i \neq j\} \quad \{p_i \vee p_j \rightsquigarrow p_i \mid w_i < w_j\}}{\{p_i \rightsquigarrow \bot \mid 1 \leq i \leq n\} \quad \{p_i \vee \alpha_i \rightsquigarrow \alpha_i \mid 1 \leq i \leq n\}}.$$

To gain some intuition about what it means that this formula is falsified in a poset recall that if the formula is falsified in $P' = (W', \leq')$ then there is some valuation V' over W' such that in the resulting model $M' = (W', \leq', V')$ we have that all the conditional that are premises of χ_P are true in M', while all the conditionals that are conclusions of χ_P are false. First, observe that then the sets $[\![p_i]\!]_{V'}$ for all $i \in \{1, \ldots, n\}$ partition W':

- The premise $\neg \bigvee_{j=1}^{n} p_j \rightsquigarrow \bot$ enforces that every world makes at least one propositional letter p_i true.
- The premises $p_i \wedge p_j \rightsquigarrow \bot$ for $i \neq j$ enforce that no world makes more than one distinct propositional letter p_i true.
- The falsity of all the conclusions $p_i \rightsquigarrow \bot$ for $i \in \{1, \ldots, n\}$ guarantees that for every i there is at least one world that makes p_i true.

Moreover, we have that:

- The truth of the premise $p_i \vee p_j \rightsquigarrow p_i$, whenever $w_i < w_j$, means that then every p_j-world in M' is above some p_i-world.
- The falsity of the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$, for all i, entails that at least one p_i-world of M' has no p_j world below it for all i and j such that $w_j \not\leq w_i$ in P.

The crucial observation, which we prove below, is that this entails that the mapping which sends every world in $[\![p_i]\!]_{V'}$ to w_i is a c-morphism from P' to P.

Lemma 2. For every finite poset P, P falsifies χ_P.

Proof. Let $P = (W, \leq)$ and $W = \{w_1, \dots, w_n\}$. Define the valuation V such that $V(p_i) = \{w_i\}$ and consider $M = (W, \leq, V)$. It is clear that then the sets $[\![p_i]\!]_V$ partition W and thus the first two premises of χ_P are true in M and the first conclusion of χ_P is false.

Also observe that if $w_i < w_j$ then $\mathrm{Min}(\leq)(\{w_i, w_j\}) = \{w_i\}$. Thus M makes the premise $p_i \vee p_j \rightsquigarrow p_i$ true.

To see that M makes the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$ false, for every fixed i, we use the alternative semantic clause from Proposition 1. Hence, we need to find some word $w \in [\![p_i \vee \alpha_i]\!]$ such that $v \not\leq w$ for all $v \in [\![\alpha_i]\!]$. Clearly we can take $w = w_i$ because every $v \in [\![\alpha_i]\!]$ is of the form $v = w_j$ for some $w_j \not\leq w_i$. □

Proposition 5. For every finite poset P', P' falsifies χ_P if and only if there exists a c-morphism f from P' to P.

Proof. The direction from right to left follows immediately from the previous lemma together with Theorem 1.

For the direction from left to right assume that $P' = (W', \leq')$ falsifies χ_P of some poset $P = (W, \leq)$ with $W = \{w_1, \dots, w_n\}$. Let V' be the valuation such that $M' \not\models \chi_P$ for $M = (P', \leq', V')$. As already described above note that this means by the truth of the first true premises that the $[\![p_i]\!]_{V'}$ partition W' and by the falsity of the first conclusion that none of the $[\![p_i]\!]_{V'}$ is empty. We then define a function $f : W' \to W$ such that $f(w') = w_i$ for the unique i such that $w' \in [\![p_i]\!]_{V'}$. We check that f is a c-morphism according to the conditions of Definition 4.1.

Condition (i): We need to show that for every $w' \in W'$ and $u \leq f(w')$ there is some $u' \leq' w'$ such that $f(u') = u$. Fix such w' and u and let $w_i = u$ and $w_j = f(w')$. Now if $u = f(w')$ then we can just let $u' = w'$ thus we can assume that $u < f(w')$, which means that $w_i < w_j$. Because then the premise $p_i \vee p_j \rightsquigarrow p_i$ is true in M' and we have $w' \in [\![p_j]\!]_{V'}$ it follows that there must be some $u' \in [\![p_i]\!]_{V'}$ with $u' \leq' w'$. From $u' \in [\![p_i]\!]_{V'}$ it follows that $f(u') = w_i = u$ and hence we have found a suitable u'.

Condition (ii): We need to show that for every $w_i \in W$ there is some $w' \in W'$ such that $f(w') = w_i$ and for every $u' \leq' w'$ we get $f(u') \leq w_i$. Fix w_i. Because the conclusion $p_i \vee \alpha_i \rightsquigarrow \alpha_i$ is false in M' it follows that there is some $w' \in [\![p_i]\!]_{V'}$ such that $v' \not\leq 'w'$ for all $v' \in [\![p_j]\!]_{V'}$ with $w_j \not\leq w_i$. Because $w' \in [\![p_i]\!]_{V'}$ we get that $f(w') = w_i$. We also get that for every $u' \leq w'$ it holds that $f(u') \leq w_i$ because otherwise we would have some $w_j = f(u')$ with $w_j \not\leq w_i$ such that $u' \in [\![p_j]\!]_{V'}$ and $u' \leq' w'$. □

We have now gathered all the ingredients for the proof of the main result of this section:

Proof of Theorem 2. The direction from left to right is a consequence of Theorem 1 which we have discussed at the end of Section 4.

For the direction from right to left assume that $\mathcal{C}$ is closed under c-morphic images. Then, let $\Gamma_{\mathcal{C}}$ be the set of all formulas valid on the class $\mathcal{C}$, that is,

$$\Gamma_{\mathcal{C}} = \{\varphi \in \mathcal{L}_1 \mid \varphi \text{ is valid in } P \text{ for all } P \text{ in } \mathcal{C}\}.$$

It is obvious from the definition all the formulas in $\Gamma_{\mathcal{C}}$ are valid in all the posets from $\mathcal{C}$. Thus, to show that Γ_C defines $\mathcal{C}$ we only need to show that if all of $\Gamma_{\mathcal{C}}$ is valid in some poset P then P is in $\mathcal{C}$. Hence, let P be such a poset. Consider its characteristic formula $\chi_P \in \mathcal{L}_1$. By Lemma 2 we know that P falsifies χ_P. Thus, it can not be the case that $\chi_P \in \Gamma_{\mathcal{C}}$, which entails that there is some poset P' in $\mathcal{C}$ such that P' falsifies χ_P. From Proposition 5 it follows that P is a c-morphic image of P' and so P must also be in $\mathcal{C}$ because the class is closed under c-morphic images. □

7 First-order correspondents

This section contains the prove that for every non-nested formula in conditional logic there is a formula in first-order logic that is true in precisely the finite posets where the formula of conditional logic is valid. Thus, every definable class of finite posets is elementary.

In this section we are assuming that all first-order formulas are formulated in a language where the relations $\leq$ and $<$ of posets and the equality relation are expressible. Or result is then formulated as follows:

Theorem 3. For all $\sigma \in \mathcal{L}_1$ one can compute a first-order formula φ_σ such that φ_σ is true in a finite poset P if and only if σ is valid in P.

Remark 2. We do not expect that Theorem 3 holds for formulas with nested conditionals. To consider an analogue of the theorem for the full language of conditional logic one would have to consider frames to be ternary relations $\leq$ such that for each world $w \in W$ the restricted relation $\leq_w$ is a poset. We conjecture that there are non-elementary classes of such frames that are definable in nested conditional logic. Applying the translation of the modal diamond $\Diamond\varphi$ as the conditional $\neg(\varphi \rightsquigarrow \bot)$ one can probably adapt examples of non-elementary classes that are definable in modal logic (Blackburn et al., 2002, sec. 3.2). The counterpart of Theorem 3 as given above for modal logic would be the claim that all frame classes that are defined by formulas in the modal language, where the modalities are not nested and propositional letters only occur in the scope of a modality, are elementary. We expect this claim to be true.

To prove Theorem 3 first observe that because of Corollary 1 it suffices to consider the case where σ is an inference of the form

$$\sigma = \frac{\Sigma}{\Gamma}.$$

We are going to define a first-order formula φ_σ that is true in some finite poset P iff σ is falsifiable in P. For the statement of the theorem we then need to consider the formula $\neg\varphi_\sigma$.

The construction of φ_σ is going to be quite complex. It might help the reader to have a look at Appendix C where we show how to apply the construction for a simple example of σ.

Let $\mathcal{A}$ the set of all propositional assignments to the propositional letters occurring in σ. It can be thought of as the set of all functions from these letters to the truth values 0 and 1. Note that $\mathcal{A}$ is a finite set. For every propositional formula $\alpha \in \mathcal{L}_0$ we define $[\![\alpha]\!] \subseteq \mathcal{A}$ to be the set of all assignments at which φ is true in the sense from classical propositional logic.

Define $\mathfrak{S} \subseteq \Gamma\Sigma^*$ to be the set of all sequences

$$(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_n \rightsquigarrow \beta_n)$$

such that:

1. $\gamma \rightsquigarrow \delta \in \Gamma$ is some conclusion from Γ.
2. $\alpha_j \rightsquigarrow \beta_j \in \Sigma$ is some premise from Σ for all $j \in \{1, \ldots, n\}$.
3. The Σ-part of the sequence does not contain repetitions. This means that $\alpha_j \rightsquigarrow \beta_j \neq \alpha_k \rightsquigarrow \beta_k$ for all $j, k \in \{1, \ldots, n\}$.

The *length* of a sequence $u \in \mathfrak{S}$ of the form $(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_n \rightsquigarrow \beta_n)$ is the number n. We consider singleton sequences $(\gamma \rightsquigarrow \delta) \in \mathfrak{S}$, where $\gamma \rightsquigarrow \delta \in \Gamma$, to be of length 0. Note that because of the last clause the length of sequences in $\mathfrak{S}$ is bounded by the number of elements in Σ. Thus $\mathfrak{S}$ is a finite set.

Let $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_n \rightsquigarrow \beta_n)$ be some sequence from $\mathfrak{S}$ and let $\alpha \rightsquigarrow \beta \in \Sigma$ be such that $\alpha \rightsquigarrow \beta \neq \alpha_i \rightsquigarrow \beta_i$ for all $i \in \{1, \ldots, n\}$. We then write $u \cdot \alpha \rightsquigarrow \beta$ to denote the sequence $(\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_n \rightsquigarrow \beta_n, \alpha \rightsquigarrow \beta) \in \mathfrak{S}$.

We then consider partial functions S from the disjoint sum $\mathfrak{S} + \mathcal{P}\Gamma$ of $\mathfrak{S}$ and the powerset of Γ to the set $\mathcal{A}$. For any such partial function S let $D_S \subseteq \mathfrak{S} + \mathcal{P}\Gamma$ be the part of the domain on which S is defined. We call a partial function S from $\mathfrak{S} + \mathcal{P}\Gamma$ to $\mathcal{A}$ *coherent* if

1. For all $\gamma \rightsquigarrow \delta \in \Gamma$ we have $(\gamma \rightsquigarrow \delta) \in D_S$ and $S((\gamma \rightsquigarrow \delta)) \in [\![\gamma \wedge \neg\delta]\!]$. Here, we consider $(\gamma \rightsquigarrow \delta)$ as a sequence in $\mathfrak{S}$ and explicitly distinguish it from the singleton set $\{\gamma \rightsquigarrow \delta\} \in \mathcal{P}\Gamma$.

2. If $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_n \rightsquigarrow \beta_n)$ and $u \in D_S$ then $S(u) \in [\![\alpha_n \wedge \beta_n]\!]$.

3. If $u = (\gamma \rightsquigarrow \delta, \alpha_1 \rightsquigarrow \beta_1, \ldots, \alpha_{n-1} \rightsquigarrow \beta_{n-1}, \alpha_n \rightsquigarrow \beta_n)$ and $u \in D_S$ then $S(u) \notin [\![\alpha_i]\!]$ for all $j \in \{1, \ldots, n-1\}$.

4. If $u \in \mathfrak{S} \cap D_S$ and $S(u) \in [\![\alpha \wedge \neg\beta]\!]$ for some $\alpha \rightsquigarrow \beta \in \Sigma$ then $u \cdot \alpha \rightsquigarrow \beta \in D_S$. Note that by the previous two conditions it is not possible that $\alpha \rightsquigarrow \beta$ is a premise that already occurs in u. Hence, $u \cdot \alpha \rightsquigarrow \beta$ is well-defined as an element of $\mathfrak{S}$.

5. If $\Delta \in D_S$ for some $\Delta \subseteq \Gamma$ then $S(\Delta) \notin [\![\alpha \wedge \neg\beta]\!]$ for all $\alpha \rightsquigarrow \beta \in \Sigma$.

Let $\mathfrak{C}$ be the set of all coherent partial functions from $\mathfrak{S} + \mathcal{P}\Gamma$ to $\mathcal{A}$. Note that $\mathfrak{C}$ is finite.

We are working with a set of first-order variables $\{x_u \mid u \in \mathfrak{S} + \mathcal{P}\Gamma\}$.

The first-order formula φ_σ is defined such that it states the existence of some coherent partial function $S \in \mathfrak{C}$ and points in the poset that for each element of D_S:

$$\varphi_\sigma = \bigvee_{S \in \mathfrak{C}} \exists x_u \ldots u, v \in D_S \ldots \exists x_v (\kappa(S) \wedge \psi(S) \wedge \chi(S) \wedge \mu_1(S) \wedge \mu_2(S)).$$

We use the notation $\exists x_u \ldots u, v \in D_S \ldots \exists x_v$ to denote a chain of existential quantifiers that contains a quantifier for every variable x_u for $u \in D_S$. The points corresponding to the elements of D_S are further constrained by the formulas $\kappa(S)$, $\psi(S)$, $\chi(S)$, $\mu_1(S)$ and $\mu_2(S)$ that all depend on S and contain only free variables of the form x_u for $u \in D_S$.

The formula $\kappa(S)$ requires that any two variables that are interpreted as the same point must also map to the same assignment under S:

$$\kappa(S) = \bigwedge_{\substack{u,v \in D_S \\ S(u) \neq S(v)}} x_u \neq x_v.$$

The formula $\psi(S)$ requires that $\Delta \in D_S$ for every $\Delta \subseteq \Gamma$ such that there is some point in W that is below all the $x_{(\zeta)}$ for $\zeta \in \Delta$. Moreover, in this case x_Δ must be below all those $x_{(\zeta)}$:

$$\psi(S) = \bigwedge_{\Delta \subseteq \Gamma} (\exists y (\bigwedge_{\zeta \in \Delta} y < x_{(\zeta)}) \rightarrow \psi'(\Delta, S)),$$

where

$$\psi'(\Delta, S) = \begin{cases} \bot, & \text{if } \Delta \notin D_S, \\ \bigwedge_{\zeta \in \Delta} x_\Delta < x_{(\zeta)}, & \text{if } \Delta \in D_S. \end{cases}$$

The formula $\chi(S)$ states that if a sequence v from D_S extends another sequence u from D_S then x_v is below x_u in the poset:

$$\chi(S) = \bigwedge_{\substack{u \in D_S \cap \mathfrak{S} \\ v = u \cdot \xi \in D_S}} x_v < x_u.$$

The formula $\mu_1(S)$ requires that $x_{(\zeta)}$ for every $\zeta = \gamma \rightsquigarrow \delta \in \Gamma$ is minimal among all those points that interpret sequences that map under S to an assignment in $[\![\gamma]\!]$:

$$\mu_1(S) = \bigwedge_{\substack{\zeta \in \Gamma, u \in D_S \\ S(u) \in [\![\gamma]\!]}} \neg x_u < x_{(\zeta)}.$$

The formula $\mu_2(S)$ requires that x_u for every extended sequence $v = v' \cdot \alpha \rightsquigarrow \beta$, where $\alpha \rightsquigarrow \beta \in \Sigma$, is minimal among all those points that map to an assignment in $[\![\alpha]\!]$:

$$\mu_2(S) = \bigwedge_{\substack{v=v'\cdot\alpha\rightsquigarrow\beta\in D_S \\ u\in D_S, S(u)\in[\![\alpha]\!]}} \neg x_u < x_v.$$

It is clear that φ_σ can be computed from the inference σ. The remaining two lemmas of this section show that φ_σ does indeed express the falsifiability of σ.

Lemma 3. If σ is falsifiable in a poset $(W, \leq)$ then $(W, \leq) \models \varphi_\sigma$.

Proof. Let $V : \mathsf{Prop} \to \mathcal{P}W$ be a valuation such that $M = (W, \leq, V)$ makes all the premises in Σ true and all the conclusion in Γ false.

To show that φ_σ is true in the poset $(W, \leq)$ we define a coherent partial function $S \in \mathfrak{C}$ and for every $u \in D_S$ an interpretation $w_u \in W$ for the variable x_u. We use $a(w) \in \mathcal{A}$ for any $w \in W$ and as a shorthand for the assignment

$$a(w) = \begin{cases} 1, & \text{if } w \in V(p), \\ 0, & \text{if } w \notin V(p). \end{cases}$$

Note that with this definition $w \in [\![\varphi]\!]_V$ iff $a(w) \in [\![\varphi]\!]$ holds for all $\varphi \in \mathcal{L}_0$.

The definition of S the selection of w_u for $u \in D_S$ proceeds in three steps that extend the domain of S. In each of the steps our choice of the w_u is such that that $S(u) = a(w_u)$ for all $u \in D_S$.

1. We first define S on sequences of the form $u = (\gamma \rightsquigarrow \delta)$ for all $\gamma \rightsquigarrow \delta \in \Gamma$. Fix a conclusion $\gamma \rightsquigarrow \delta \in \Gamma$. Because $\gamma \rightsquigarrow \delta$ is false in M it follows that there is some world w that is minimal in $[\![\gamma]\!]_V$ such that $w \notin [\![\delta]\!]_V$. We let $w_u = w$ be this world and define $S(u) = a(w)$. Note that this ensures that S satisfies condition 1 from the definition of coherent functions.

2. In this step we inductively extend the definition of S to longer and longer sequences from $\mathfrak{S}$ such that condition 4 becomes satisfied. This definition is by induction on the length of sequences $u \in \mathfrak{S}$. In every step, where we add a sequence $u = v \cdot \alpha \rightsquigarrow \beta$, we ensure that

   (a) S and u satisfy conditions 2 and 3 for S being coherent,

   (b) $w_u \leq w_v$, and

   (c) $w_u \in \mathrm{Min}([\![\alpha]\!]_V)$.

   The base case consists simply of all the sequences of length 0 that were added in the previous step. In the inductive step assume that we have already added all required sequences of length n. Let $v \in D_S$ be a sequence of length n and assume that $S(v) \in [\![\alpha \wedge \neg\beta]\!]$ for some $\alpha \rightsquigarrow \beta \in \Sigma$. We are going to add the sequence $u = v \cdot \alpha \rightsquigarrow \beta$ to the definitional domain of S. Because $S(v) = a(w_v)$ it follows that $w_v \in [\![\alpha \wedge \neg\beta]\!]_V$. Since $M \models \alpha \rightsquigarrow \beta$ there must be some $w \in \mathrm{Min}([\![\alpha]\!]_V)$ with $w \in [\![\beta]\!]_V$ and $w < w_v$. We set $w_u = w$ and $S(u) = a(w_u)$. This takes care of items 2b and 2c for u. This definition also satisfies condition 2 for S being coherent because $S(u) \in [\![\alpha \wedge \beta]\!]$.

   To check that u satisfies condition 3 assume that $v' \cdot \alpha' \rightsquigarrow \beta'$ is a proper initial segment of u. We need to show that $S(u) \notin [\![\alpha']\!]$, or equivalently that $w_u \notin [\![\alpha']\!]_V$. Inductively we can assume that all initial segments of v already satisfy item 2b from above. Thus, $w_v \leq w_{v'}$ and together with $w_u < w_v$ we obtain $w_u < w_{v'}$. Moreover, because v' satisfies item 2c we know that $w_u \notin [\![\alpha']\!]_V$. Combining these facts we obtain that $w_{v'} \in \mathrm{Min}([\![\alpha']\!]_V)$.

3. Lastly, we consider any subset $\Delta \subseteq \Gamma$ such that there exists some $u_\Delta \in W$ such that $u_\Delta < w_{(\gamma \rightsquigarrow \delta)}$ for all $\gamma \rightsquigarrow \delta \in \Delta$. Fix such a Δ, define w_Δ to be any minimal element of $(W, \leq)$ that is below u_Δ and set $S(\Delta) = a(w_\Delta)$. To see that this definition satisfies condition 5 on

coherent partial functions we need to see that $w_\Delta \notin [\![\alpha \wedge \neg\beta]\!]_V$ for all $\alpha \rightsquigarrow \beta \in \Sigma$. If this was not the case then we would have that w_Δ is minimal in $[\![\alpha]\!]_V$, as it is minimal in W, but $w_\Delta \notin [\![\beta]\!]_V$. This would contradict the assumption that the model M makes the premise $\alpha \rightsquigarrow \beta \in \Sigma$ true.

It is clear that the partial function S that is defined in this way is coherent. It remains to be seen that the disjunct of φ_σ that corresponds to S is true in $(W, \leq)$. To this aim we interpret that existential variable x_u as the element $w_u \in W$ for all $u \in D_S$. Because we have that $S(u) = a(w_u)$ for all $u \in D_S$ it is guaranteed that $\kappa(S)$ is true with this assignment. In the third step of the construction of S we make sure that $\psi(S)$ is true in $(W, \leq)$. The formula $\chi(S)$ holds because of item 2b from the second step. In the first step we chose $w_{(\gamma \rightsquigarrow \delta)} \in \mathrm{Min}([\![\gamma]\!]_V)$ and hence $\mu_1(S)$ is true in $(W, \leq)$. Lastly, $\mu_2(S)$ holds because of item 2c from the second step. □

Lemma 4. If $(W, \leq) \models \varphi_\sigma$ then σ is falsifiable in the poset $(W, \leq)$.

Proof. Assume that the first-order formula φ_σ is true in the poset $(W, \leq)$. This means that there is some good $S \in \mathfrak{C}$ such that $\kappa(S)$, $\psi(S)$, $\chi(S)$, $\mu_1(S)$ and $\mu_2(S)$ hold for some interpretation of the existential variables from $\{x_u \mid u \in D_S\}$ in $(W, \leq)$. For all $u \in D_S$ let w_u be the value of the variable x_u for which this is the case. Define $X \subseteq W$ to be the set $X = \{w_u \in W \mid u \in D_S\}$.

Note that because $\kappa(S)$ holds for this interpretation of the existential variables it follows that $S(u) = S(v)$, whenever $w_u = w_v$ for some $u, v \in D_S$. For this reason the following function is well-defined $s : X \to \mathcal{A}, w_u \mapsto S(u)$.

Our next goal is to define a function $f : W \to X$ from which we then define the valuation $V : \mathsf{Prop} \to \mathcal{P}W$ by setting

$$V(p) = \{w \in W \mid s(f(w))(p) = 1\}.$$

To define the value of $f(w) \in X$ for some $w \in W$ we distinguish cases depending on how w is situated relative to the elements in X.

1. If there is some $v \in X$ such that $v \leq w$ then we let $f(w) = y$ for some chosen $y \in X$ that is maximal among all $z \in X$ with $z \leq w$. Such a maximal y always exists because X is finite.

2. If there is no $v \in X$ such that $v \leq w$ then we consider the set $\Delta = \{\zeta \in \Gamma \mid w < w_{(\zeta)}\}$. Because $\psi(S)$ holds of our assignment of variables we have that $\Delta \in D_S$. Thus, we can set $f(w) = w_\Delta \in X$.

Note that because of the first clause f is the identity on all $w \in X \subseteq W$.

It remains to be proven that $M = (W, \leq, V)$ makes all conditionals in Σ true and all conditionals in Γ false.

Thus, consider any premise $\alpha \rightsquigarrow \beta \in \Sigma$ and world $w \in W$. To show that $M \models \alpha \rightsquigarrow \beta$ we use the reformulation of the semantic clause from Proposition 1. Thus consider any $w \in [\![\alpha]\!]_V$. We need to find a $w' \leq w$ with $w' \in [\![\alpha \wedge \beta]\!]_V$. Distinguish cases depending on the definition of $f(w)$.

First consider the case where there is no $v \in X$ such that $v \leq w$. Then $f(w) = w_\Delta$ for some $\Delta \subseteq \Gamma$. Because of condition 5 of coherence it holds that $w_\Delta \notin [\![\alpha \wedge \neg\beta]\!]_V$. Note that the definition of V is such that w satisfies the same propositional letters as w_Δ because $f(w) = w_\Delta$. Thus $w \notin [\![\alpha \wedge \neg\beta]\!]_V$. Because $w \in [\![\alpha]\!]_V$ it follows that $w \in [\![\alpha \wedge \beta]\!]_V$ and we can take $w' = w$.

In the other case there is some $v \in X$ with $v \leq w$ then let $y \in X$ be such that $y = f(w)$ and $y \leq w$. Because $y \in X$ we have that $y = w_u$ for some $u \in \mathfrak{S} + \mathcal{P}\Gamma$. We distinguish further cases depending on whether $u \in \mathcal{P}\Gamma$ or $u \in \mathfrak{S}$. If $u \in \mathcal{P}\Gamma$ then $u = \Delta$ for some $\Delta \subseteq \Gamma$ and we can reason precisely as in the previous case. In the other case we have that $u \in \mathfrak{S}$. From $f(w) = w_u$ and $w \in [\![\alpha]\!]_V$ it follows that $w_u \in [\![\alpha]\!]_V$, because w and w_u satisfy the same propositional letters under V. Note that we can assume that $w_u \in [\![\alpha \wedge \neg\beta]\!]_V$ because if $w_u \in [\![\alpha \wedge \beta]\!]_V$ then also $w \in [\![\alpha \wedge \beta]\!]_V$ and we can set $w' = w$. But $w_u \in [\![\alpha \wedge \neg\beta]\!]_V$ means that $S(u) = s(w_u) \in [\![\alpha \wedge \neg\beta]\!]$. By condition 4 from the definition of coherency this entails that $v = u \cdot \alpha \rightsquigarrow \beta \in D_S$. From condition 2 we get that $S(v) \in [\![\alpha \wedge \beta]\!]$ and thus $w_v \in [\![\alpha \wedge \beta]\!]_V$. Using that $\chi(S)$ is true in $(W, \leq)$ we have that $w_v \leq w_u$. Using $w_u = y \leq w$ it follows that $w_v \leq w$. Thus we can take $w' = w_v$.

Lastly, we argue that the conclusions are false in M. Consider any conclusion $\gamma \rightsquigarrow \delta \in \Gamma$. By condition 1 on the coherent function S we have that $S(\gamma \rightsquigarrow \delta) \in [\![\gamma \wedge \neg\delta]\!]$. Thus $w_{(\gamma \rightsquigarrow \delta)} \in [\![\gamma \wedge \neg\delta]\!]_V$. Because we can show that for all $w \in W$ with $w < w_{(\gamma \rightsquigarrow \delta)}$ we have $w \notin [\![\delta]\!]_V$ it follows by the alternative formulation of the semantics in Proposition 1 that $M \not\models \gamma \rightsquigarrow \delta$. To this aim consider any $w < w_{(\gamma \rightsquigarrow \delta)}$. We distinguish cases depending on the clause defining $f(w)$.

If there is some $v \in X$ with $v \leq w$ then consider the $y \in X$ with $y \leq w$ such that $f(w) = y$. By transitivity it follows that $y \leq w_{(\gamma \rightsquigarrow \delta)}$ and hence we can use that $\mu_1(S)$ is true to derive that $y \notin [\![\delta]\!]_V$. Because $f(w) = y$ we have that the valuation V is the same with respect to w as with respect to w and thus also $w \notin [\![\delta]\!]_V$.

If there is no $v \in X$ with $v \leq w$ then $f(w) = w_\Delta$ for some $\Delta \subseteq \Gamma$ such that $\gamma' \rightsquigarrow \delta' \in \Delta$ whenever $w < w_{(\gamma' \rightsquigarrow \delta')}$. Because $w < w_{(\gamma \rightsquigarrow \delta)}$ this means that $\gamma \rightsquigarrow \delta \in \Gamma$. Because $\psi(S)$ is true in $(W, \leq)$ it follows that $w_\Delta < w_{(\gamma \rightsquigarrow \delta)}$ and because $\mu_1(S)$ is true it follows that $w_\Delta \notin [\![\delta]\!]_V$. Because $w_\Delta = f(w)$ we can conclude that $w \notin [\![\delta]\!]_V$. □

8 Conclusion

In this paper we provide some results on frame definability in conditional logic. We show that definable classes of posets are characterized by being closed under c-morphic images and we provide an algorithm that shows that every definable class of posets is elementary.

An obvious direction for further research is to lift some of the limitations of our setting. First, one might be interested in studying definability of ternary relations by formulas in full conditional logic, where the conditional can occur nested. We expect that to obtain results in this direction one would have to combine ideas from this paper with ideas from the work on frame definability for normal modal logic. Second, one might try to generalize to the infinite case. We conjecture that most of our results generalize to wellfounded orders. For non-wellfounded order, however, the example in Appendix A suggests that the theory of frame definability is quite different than in the finite case.

A further interesting open question is whether there are any general completeness results for conditional logics. Many of the examples of formulas that we give in Section 3 were taken from literature that proves completeness results for the logic that is axiomatized by these formulas. One might hope that there is a general completeness result that characterizes a class of formulas such that, if one adds such a formula as an additional axiom to the logic of Burgess (1981) and Veltman (1985), then one obtains a logic that is complete for the class of posets that the formula defines.

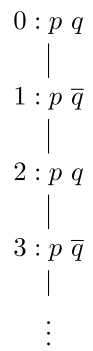
A Definability over non-wellfounded orders

In this first part of the appendix we provide an example which shows that in over non-wellfounded infinite orders frame definability no longer behaves as one would expect from the finite case. The following semantic clause has been suggested by Burgess (1981) to evaluate the conditional in a potentially non-wellfounded infinite model $M = (W, \leq, V)$:

$$M \models \varphi \rightsquigarrow \psi \quad \text{iff} \quad \text{for all } w \in [\![\varphi]\!]_V \text{ there is a } v \in [\![\varphi]\!])_V \text{ with } v \leq w \text{ such that for all } u \leq v \text{ if } u \in [\![\varphi]\!]_V \text{ then } v \in [\![\varphi]\!]_V.$$

It can be shown that over wellfounded orders this clause is equivalent to the minimization clause from Section 2.2. In orders that are not wellfounded this clause is better behaved than the minimization clause because not every non-empty subset of a non-wellfounded order has minimal elements.

Recall from Example 3 that over finite posets the formula $(p \rightsquigarrow q) \vee (p \rightsquigarrow \neg q)$ define the class of linear orders. This is no longer the case over non-wellfounded posets. To see this consider the following preferential model $M = (\omega, \geq, V)$:



Here, the poset $(\omega, \geq)$ is the converse of the standard order over the natural numbers, p is true at all numbers and q is true at a number iff it is even. Clearly, this is a linear order but according to the above semantic clause the model M does not satisfy the formula $(p \rightsquigarrow q) \vee (p \rightsquigarrow \neg q)$. Thus, in the general, non-wellfounded case $(p \rightsquigarrow q) \vee (p \rightsquigarrow \neg q)$ does not define the class of linear orders.

B More examples of definable classes

In this part of the appendix we discuss more complex examples of conditionally definable classes of finite posets.

B.1 Chains of length at most n

The class of posets that have at most a chain of length n can be defined through the rule

$$\frac{\{\varphi_{i+1} \rightsquigarrow \neg\varphi_i \mid i \in [n]\}}{p_0 \rightsquigarrow \bot}$$

where $\varphi_i := \bigvee_{j=0}^{i} p_j$.

In fact, assume that P is a poset with a $n+1$-chain, i.e. there exists a sequence $x_n < \ldots < x_0$. Then, under the valuation $V(p_i) = \{x_i\}$, $i \in [n]$, it happens that

$$\mathsf{min}\{[\![\varphi_i]\!]\} = [\![p_i]\!] = \{x_i\}$$

thus, $\varphi_{i+1} \rightsquigarrow \neg\varphi_i$ for each $i \in [n]$, but $\mathsf{min}\{p_0\} = \{x_0\} \not\subseteq \emptyset$.

Viceversa, assume by contradiction that P has at most an n-chain, each $\varphi_{i+1} \rightsquigarrow \neg\varphi_i$ is satisfied but $p_0 \rightsquigarrow \bot$ is not satisfies, i.e. there exists a p_0 world, let's call it x_0. Because of the premises, it must be that every φ_{i+1}-minimal world y is a $\bigwedge_{j=0}^{i} \neg p_j$ world and moreover is must satisfy p_{i+1}. Thus, x_0 cannot be φ_1-minimal, and therefore it exists $x_1 > x_0$ which is p_1-minimal and it's not a p_0-world. Again, x_1 cannot be a φ_2-minimal, leading to the existence of an x_2 which is a p_2-world but neither p_1 nor p_0-world. With the same argument we can prove the existence of a $n+1$-chain, reaching a contradiction.

B.2 At most n-minimal elements

Before we give the rule to define posets with at most n minimal elements, we will first introduce some notation. In the rule, we will use n different propositional letters $P_n = \{p_1, \ldots, p_n\}$. For any subset $a \subseteq P_n$, let $\alpha_n(a) = \bigwedge_{p_i \in a} p_i \wedge \bigwedge_{p_i \notin a} \neg p_i$. For any integer m, define $\varphi_n(a_1, \ldots, a_m) = \bigvee_{i=1}^{m} \alpha_n(a_i)$.

Thus, the class of posets with at most n minimal elements is defined by the rule:

$$\frac{}{\mathcal{B}(\top, P_n)}$$

where $\mathcal{B}(\top, P_n)$ stands for the following set of formulas:

$$\{\top \rightsquigarrow \varphi_n(a_1, \dots, a_n) \mid a_1, \dots, a_n \in \mathcal{P}P_n\}.$$

Now, if P has at most n elements, then the set of formulas $\mathcal{B}(\top, P_n)$ denotes all the possible combinations of valuations that those elements can have, thus P satisfies the rule.

Viceversa, if P has $m > n$ minimal elements it is enough to mark differently each minimal element, which is always doable because the possible markings are $2^n > n$ and n is a non-zero natural number. Thus, under this valuation none of the conclusions is satisfied because every conclusion allows at most n markings.

B.3 n-Weak orders

In this part we discuss a generalization of the conditions on semitransitive orders, which we call n-weak orders.

Definition B.1. A poset $P = (W, \leq)$ is called *n-weak* if for all $x_0, \dots, x_n, y \in W$ such that $x_0 < \dots < x_n$ we get that either $x_0 < y$ or $y < x_n$.

With this definition we have that the 1-weak orders are exactly the weak orders and the 2-weak orders are exactly the semitransitive orders.

Proposition 6. For every natural number $n > 2$, the class of n-weak orders is defined by the following rule:

$$\frac{p \wedge \bigwedge_{j=0}^{n-2} \neg q_j \rightsquigarrow \neg q_{n-1} \qquad \left[p \wedge \bigwedge_{j=0}^{i} \neg q_j \rightsquigarrow q_{i+1} \right]_{i=0}^{n-3} \qquad p \rightsquigarrow q_0}{p \rightsquigarrow \neg q_n \qquad p \wedge q_{n-1} \rightsquigarrow \bigvee_{j=0}^{n-2} q_j}.$$

Notice that the middle premises are not defined in case $n = 2$ (compare to the rule for semitransitivity from Figure 3).

Proof. We first show that any n-weak order validates this rule. Assume by contraposition that the rule is not valid in some poset P, i.e. there is a valuation V such that all the rule's premises are satisfied but there is an element y which is p-minimal but such that $y \in [\![q_{n-1}]\!]$ and some element x_n which is $p \wedge q_{n-1}$-minimal but such that $x_n \notin [\![q_0 \vee \dots \vee q_{n-2}]\!]$. Notice this already implies that $y \neq x_n$ since q_0 must be true at y as by assumption the premises of the rule are satisfied. We show that P is not a n-weak order. Observe that $x_n \in [\![p \wedge \neg q_0 \wedge \dots \wedge \neg q_{n-2}]\!]$ but $x_n \notin [\![\neg q_{n-1}]\!]$, hence it cannot be $p \wedge \neg q_0 \wedge \dots \wedge \neg q_{n-2}$-minimal. This means that there must be some $x_{n-1} < x_n$ which is $p \wedge \neg q_0 \wedge \dots \wedge \neg q_{n-2}$-minimal and such that $x_n \in [\![\neg q_{n-1}]\!]$. However, observe that this contradicts the second premise in the $i = n-3$ case, hence x_{n-1} cannot be $p \wedge \neg q_0 \wedge \dots \wedge \neg q_{n-3}$-minimal. Again, this implies the existence of some $x_{n-2} < x_{n-1}$ which is $p \wedge \neg q_0 \wedge \dots \wedge \neg q_{n-3}$-minimal and such that $x_{n-2} \in [\![q_{n-2}]\!]$. By an inductive argument there is a chain of elements $x_0 < x_1 < \dots < x_{n-2} < x_{n-1} < x_n$ such that, for any $0 \leq i \leq n-2$, we have that $x_i \in [\![p \wedge q_i]\!]$ and $x_i \in [\![\neg q_0 \wedge \dots \wedge \neg q_{i-1}]\!]$. Moreover, for any $0 \leq i \leq n-1$, we also have that $x_i \notin [\![q_{n-1}]\!]$, as p is true at all these elements but x_n is $p \wedge q_{n-1}$-minimal. To conclude, it suffices to notice that neither $x_0 < y$ nor $y < x_n$ can be true, for the former contradicts the p-minimality of y whereas the latter the $p \wedge q_{n-1}$-minimality of x_n. Thus, P is not a n-weak order.

Vice versa, let $P = (W, \leq)$ be a poset which is not an n-weak order. Then, there must be $x_0, \dots, x_{n+1}, y \in W$ such that $x_0 < \dots < x_{n+1}$ but $x_0 \not< y$ as well as $y \not< x_{n+1}$. Let $X = \{x_0, \dots, x_{n+1}\}$. Consider a valuation V defined as follows:

$$\begin{aligned} V(p) &= X \cup \{y\} \\ V(q_{n-1}) &= \{y, x_n\} \\ V(q_i) &= \{x_i\} \qquad \text{for each } 1 \leq i \leq n-2 \\ V(q_0) &= \{y, x_0\} \end{aligned}$$

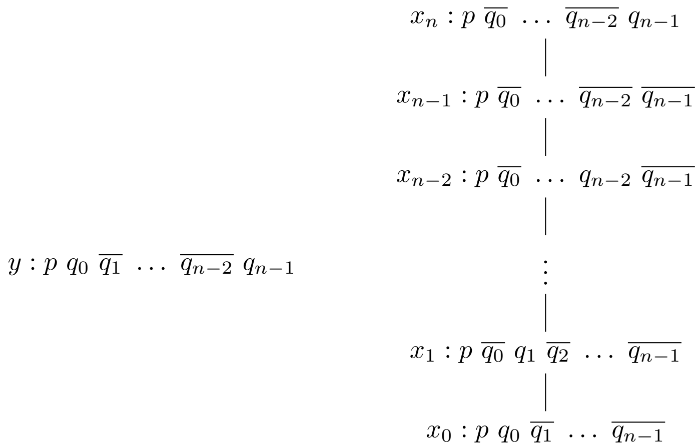
Then, it holds:

$$\min\{[\![p]\!]\} = \min\{[\![q_0]\!]\} = \{y, x_0\} \quad \min\{[\![p \wedge q_{n-1}]\!]\} = \{y, x_n\}$$
$$\min\{[\![p \wedge \bigwedge_{j=0}^{n-2} \neg q_j]\!]\} = \{x_{n-1}\}$$

and, for every $1 \leq i \leq n-2$:

$$\min\{[\![p \wedge \bigwedge_{j=0}^{i-1} \neg q_j]\!]\} = \{x_i\}$$

Therefore, all the premises are satisfied but none of the conclusions hold. The following picture summarizes the proof:



□

C Computing a first-order correspondent

In this part of the appendix we apply the construction from Section 7 to compute the first-order correspondent φ_σ for the case where σ is the inference

$$\frac{\top \rightsquigarrow p}{\neg p \rightsquigarrow \bot.}$$

We have already seen in Example 1 that this inference defines the class of all antichains. Thus, φ_σ should become a first-order formula that expresses that the poset is not an antichain.

Because σ only contains the propositional letter p we get that $\mathcal{A}$ contains only two elements: One assignment which makes p true and one assignment which makes p false. We write these assignments simply as p and $\overline{p}$. It is easy to see from the definition that in this example the set $\mathfrak{S}$ contains the following two elements:

$$u_0 = (\neg p \rightsquigarrow \bot)$$
$$u_1 = (\neg p \rightsquigarrow \bot, \top \rightsquigarrow p).$$

Let us then see how we can describe the coherent partial functions from S to $\mathfrak{S} + \mathcal{P}\Gamma$, where $\Gamma = \{\neg p \rightsquigarrow \bot\}$. First, observe that by condition 1 in the definition of coherence we have for each coherent S that $u_0 \in D_S$ and $S(u_0) = \overline{p}$. By condition 4 it follows that also $u_1 \in D_S$ and we must have that $S(u_1) = p$ by condition 2. Just from the definition of coherence we can not deduce anything about which elements $\Delta \subseteq \Gamma$ are in the domain of S. However, it follows from condition 5 that if $\Delta \in D_S \cap \mathcal{P}\Gamma$ then $S(\Delta) = p$.

In the formula φ_σ we then have a disjunct for every coherent function $S \in \mathfrak{C}$. We can exclude some functions S from our calculation because the associated disjunct is equivalent to $\bot$.

We can exclude all functions S with $\emptyset \notin D_S$ because if $\emptyset \notin D_S$ then $\psi(S) = \exists y \top \to \bot$ which is equivalent to $\bot$. Thus, the disjunct for such S is also equivalent to $\bot$ and as no influence on the meaning of φ_σ.

We can also discard the case where $\Gamma \notin D_S$. In this case the disjunct of φ_σ for S contains the conjunct $\psi(s) = \exists y(y < x_{u_0}) \to \bot$, which is equivalent to $\neg\exists y(y < x_{u_0})$. This however contradicts the other conjunct $\chi(S) = x_{u_1} < x_{u_0}$. It follows that the whole disjunct for S is equivalent to $\bot$.

It follows from all these considerations that the only disjunct from φ_σ that does not trivialize is the one for the coherent function S with $D_S = \{u_0, u_1, \emptyset, \Gamma\}$ such that $S(u_0) = \overline{p}$, $S(u_1) = p$, $S(\emptyset) = p$ and $S(\Gamma) = p$. The conjuncts of the disjunct for S then look as follows:

$$\begin{aligned}
\kappa(S) &= x_{u_0} \neq x_{u_1} \wedge x_{u_0} \neq x_\emptyset \wedge x_{u_0} \neq x_\Gamma \\
\psi(S) &= (\exists y \bot \to \bot) \wedge (\exists y(y < x_{u_0}) \to x_\Gamma < x_{u_0}) \equiv \exists y(y < x_{u_0}) \to x_\Gamma < x_{u_0} \\
\chi(S) &= x_{u_1} < x_{u_0} \\
\mu_1(S) &= \neg x_{u_0} < x_{u_0} \equiv \top \\
\mu_2(S) &= \neg x_{u_1} < x_{u_1} \wedge \neg x_\emptyset < x_{u_1} \wedge \neg x_\Gamma < x_{u_1} \equiv \neg x_\emptyset < x_{u_1} \wedge \neg x_\Gamma < x_{u_1}.
\end{aligned}$$

The indicated equivalences are with respect to the class of all posets, they are not necessarily logical equivalences in first-order logic. Putting all of these together, we have that the disjunct for S, and thus also φ_σ, is equivalent to

$$\begin{aligned}
&\exists x_{u_0} \exists x_{u_1} \exists x_\emptyset \exists x_\Gamma (x_{u_0} \neq x_{u_1} \wedge x_{u_0} \neq x_\emptyset \wedge x_{u_0} \neq x_\Gamma \wedge (\exists y(y < x_{u_0}) \to x_\Gamma < x_{u_0}) \\
&\qquad\qquad \wedge x_{u_1} < x_{u_0} \wedge \neg x_\emptyset < x_{u_1} \wedge \neg x_\Gamma < x_{u_1})
\end{aligned}$$

Clearly, the conjunct $x_{u_1} < x_{u_0}$ implies the antecedent of the implication $\exists y(y < x_{u_0}) \to x_\Gamma < x_{u_0}$. So the whole formula simplifies to

$$\begin{aligned}
&\exists x_{u_0} \exists x_{u_1} \exists x_\emptyset \exists x_\Gamma (x_{u_0} \neq x_{u_1} \wedge x_{u_0} \neq x_\emptyset \wedge x_{u_0} \neq x_\Gamma \wedge x_\Gamma < x_{u_0} \\
&\qquad\qquad \wedge x_{u_1} < x_{u_0} \wedge \neg x_\emptyset < x_{u_1} \wedge \neg x_\Gamma < x_{u_1})
\end{aligned}$$

Over posets $x < y$ implies $x \neq y$ and hence we can simplify further:

$$\exists x_{u_0} \exists x_{u_1} \exists x_\emptyset \exists x_\Gamma (x_{u_0} \neq x_\emptyset \wedge x_\Gamma < x_{u_0} \wedge x_{u_1} < x_{u_0} \wedge \neg x_\emptyset < x_{u_1} \wedge \neg x_\Gamma < x_{u_1})$$

We also have that in posets because of transitivity and irreflexivity $x_\Gamma < x_{u_0} \wedge x_{u_1} < x_{u_0}$ implies $\neg x_\Gamma < x_{u_1}$:

$$\exists x_{u_0} \exists x_{u_1} \exists x_\emptyset \exists x_\Gamma (x_{u_0} \neq x_\emptyset \wedge x_\Gamma < x_{u_0} \wedge x_{u_1} < x_{u_0} \wedge \neg x_\emptyset < x_{u_1})$$

Lastly, observe that over posets this formula is equivalent to

$$\exists x_{u_0} \exists x_{u_1} (x_{u_1} < x_{u_0}),$$

because we can interpret $x_\emptyset$ and x_Γ with the same object as x_{u_1}. Clearly, this formula is true in precisely those posets that are not an antichain.

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