Uniform Interpolation for Coalgebraic Fixpoint Logic

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Abstract

We use the connection between automata and logic to prove that a wide class of coalgebraic fixpoint logics enjoys uniform interpolation. To this aim, first we generalize one of the central results in coalgebraic automata theory, namely closure under projection, which is known to hold for weak-pullback preserving functors, to a more general class of functors, i.e., functors with quasi-functorial lax extensions. Then we will show that closure under projection implies definability of the bisimulation quantifier in the language of coalgebraic fixpoint logic, and finally we prove the uniform interpolation theorem.

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1 Introduction

The connection between automata and logic goes back to the early seventies by the works of Büchi [3] and Elgot [6], who showed that finite automata and monadic second-order logic have the same expressive power over finite words, and that the transformations from formulas to automata and vice versa are effective. This connection has found important applications and landmark results, such as Rabin's decidability theorem [18]. During the last twenty years study of the link between automata and logic has been continued and many interesting results have been obtained, such as results in [10], where Janin and Walukiewicz established the connection between the modal μ-calculus and parity automata operating on labeled transition systems.

The coalgebraic perspective on the link between automata and logic has been uniformly studied in [22], where the author introduces the notion of a coalgebra automaton and establishes the connection between these automata and coalgebraic fixpoint logic based on Moss' modality ∇ [15]. Coalgebraic fixpoint logic is a powerful extension of coalgebraic modal logic [15] with fixpoint operators. The main contribution of this paper will be to add *uniform interpolation* to the list of properties of coalgebraic fixpoint logic.

A logic has *interpolation* if, whenever we have formulas a and b such that $\models a \to b$ (meaning that the formula $a \to b$ holds in every model), then there is an *interpolant* formula c in the *common language* of a and b (i.e., c may use only propositional letters that appear both in a and b), such that $\models a \to c$ and $\models c \to b$. This notion is familiar from first-order logic, and is known there as Craig interpolation [4] . Some logics enjoy a much stronger

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version of interpolation, namely uniform interpolation, which has been introduced by Pitts in [17]. A logic has uniform interpolation if the interpolant c does not really depend on b itself, but only on the language that b shares with a. Although it is easy to show that classical propositional logic has uniform interpolation, not many logics have this property, for instance first-order logic has interpolation, but it does not enjoy the uniform version [9].

As a motivation for studying uniform interpolation, let us mention some recent works on this property. Starting with the seminal work of Pitts [17] who introduced this version of interpolation and proved that intuitionistic logic has uniform interpolation, the study of this property for different logics has been actively pursued by various authors. In modal logic, Shavrukov [21] proved that the Gödel-Löb logic **GL** has uniform interpolation. Subsequently, Ghilardi [7] and Visser [23] independently established the property for modal logic **K**, while [8] contains negative results for modal logic **S4**. In the theory of modal fixpoint logic D'Agostino and Hollenberg proved that the modal μ-calculus has uniform interpolation [5].

In this paper we study uniform interpolation in the context of coalgebraic fixpoint logic. More specifically, we restrict attention to set functors T that preserve finite sets and that admit a so-called quasi-functorial lax extension L, that is, a certain kind of relation lifting satisfying somewhat weaker conditions than the standard Barr extension (see Definition 2.11 for the details). This class of functors includes the ones that preserve weak pullbacks, but also the monotone neighborhood functor, and it is closed under various natural operations on functors, see Fact 2.15. For each of these functors we consider a coalgebraic modal fixpoint logic $\mu\mathcal{L}^{\mathsf{T}}_{L}$, in the style of [22], where the semantics of the Moss-style modality is given by the relation lifting L (a definition is given in section 3). Our main result, Theorem 5.3 states that the resulting logic enjoys the property of uniform interpolation; our proof also applies to the fixpoint-free fragment of the logic.

As usual in the setting of modal logic, our proof is based on the link between uniform interpolation and the definability of a certain nonstandard second-order quantifier, the so-called *bisimulation quantifier*. More specifically, our aim will be to define, for each proposition letter p, a map $\exists p$ on $\mu\mathcal{L}^{\mathsf{T}}_{L}$, and prove that this map satisfies

$$\mathbb{S}, s \Vdash \exists p.b \quad \text{iff} \quad \mathbb{S}', s' \Vdash b, \text{ for some } \mathbb{S}', s' \text{ with } \mathbb{S}, s \leftrightarroweq_p \mathbb{S}', s', \tag{1}$$

for all pointed coalgebras $(\mathbb{S}, s)$. Here $\leftrightarroweq_p$ denotes bisimilarity, with respect to the relation lifting L, up to the proposition letter p.

Our proof follows the automata-theoretic approach by D'Agostino and Hollenberg. That is, in section 3.2 we define a class of nondeterministic parity automata that closely correspond to our language, in the sense that there are effective translation transforming a $\mu\mathcal{L}^{\mathsf{T}}_{L}$-formula into an equivalent automaton, and vice versa. Our main technical result, generalizing earlier work by Kupke and the third author [12], is in section 4; it provides a construction revealing that the class of coalgebra automata associated with our logic, is closed under projection. From this we may easily derive the definability of bisimulation quantifiers in our logic.

In order to finish the introduction, let us mention some related work. First of all, our paper should be considered as the publication of results from the first author's MSc thesis [13] on uniform interpolation for the fixpoint-free fragment of our language. Pattinson [16] introduced a variant of Moss-style coalgebraic modal logic which nicely works for all set functors: the so-called logic of exact covers, and he showed that this logic enjoys uniform interpolation. Since the modality of this language seems to be inherently non-monotonic, it is not so clear how to extend his result to the setting with fixpoint operators.

Overview. We first fix notation and terminology and equip the reader with the necessary background material. In section 3 we introduce coalgebraic fixpoint logic and give a brief introduction to automata theory. After that, we prove in section 4 our main technical result. We show that if functor $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ has a quasi-functorial lax extension L which preserves diagonals, then T-automata are closed under projection. Finally in section 5 we combine the results from section 3 and section 4 in order to prove uniform interpolation for coalgebraic fixpoint logic $\mu\mathcal{L}_L^{\mathsf{T}}$. We finish the paper in section 6 with an outlook on future results.

2 Preliminaries

This section contains some of the preliminaries and fixes the notation. We presuppose that reader has made contact with basic concepts from category theory before. For example we assume familiarity with basic notions such as categories, functors, natural transformations and isomorphic categories.

We also presupposes knowledge of the theory of coalgebras. An extensive introduction is given for example in [19].

2.1 Set Functors

We will work in the category Set, that has sets as objects and functions as arrows. For sets $X' \subseteq X$, the inclusion map from X' to X is denoted by $i_{X',X} : X' \hookrightarrow X$, $x \mapsto x$. For a function $f : X \to Y$ we define the set $Rng(f) = \{y \in Y \mid \exists x \in X, f(x) = y\} \subseteq Y$. In the following we assume, if not explicitly stated otherwise, that functors are covariant endofunctors in the category Set.

We first introduce some of the functors that concern us in this paper. The *powerset functor* is the functor $\mathcal{P} : \mathsf{Set} \to \mathsf{Set}$, which maps a set S to the set of all its subsets $\mathcal{P}S = \{V \mid V \subseteq S\}$. A function $f : S \to T$ is mapped to $\mathcal{P}f : \mathcal{P}S \to \mathcal{P}T$, which is defined for any $V \subseteq S$ by $\mathcal{P}f(V) = f[V] = \{f(v) \mid v \in V\}$. The *contravariant powerset* functor $\breve{\mathcal{P}}$ also maps a set S to $\breve{\mathcal{P}}S = \mathcal{P}S$. On functions $\breve{\mathcal{P}}$ is the inverse image map, that is for an $f : S \to T$ we have $\breve{\mathcal{P}}f : \breve{\mathcal{P}}S \to \breve{\mathcal{P}}T$, $V \mapsto f^{-1}[V]$. The *neighborhood* functor $\mathcal{N} = \breve{\mathcal{P}}\breve{\mathcal{P}}$ is the double contravariant powerset functor. Given a set S and an element $\alpha \in \mathcal{N}S$, we define

$$\alpha^{\uparrow} := \{X \in \mathcal{P}S \mid Y \subseteq \ X \text{ for some } Y \in \alpha\}$$

and we say that α is *upward closed* if $\alpha = \alpha^{\uparrow}$. The *monotone neighborhood* functor $\mathcal{M}$ is the restriction of the neighborhood functor to upward closed sets. More concretely the functor $\mathcal{M}$ is given by $\mathcal{M}S := \{\beta \in \mathcal{N}S \mid \beta \text{ is upward closed}\}$, while for $f : S \to T$, we define $\mathcal{M}f : \mathcal{M}S \to \mathcal{M}T$ by $\mathcal{M}f(\beta) := (\mathcal{N}f(\beta))^{\uparrow}$.

▶ **Definition 2.1.** $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$ *preserves inclusions* if $\mathsf{T}i_{A,B} = i_{\mathsf{T}A,\mathsf{T}B}$ for all sets $A \subseteq B$.

▶ **Proposition 2.2.** *If* $\mathsf{T} : Set \to Set$ *preserves inclusions, then* $\mathsf{T}(Rng(f)) = Rng\mathsf{T}(f)$ *for any function* f *in* Set.

In coalgebraic logic one pays special attention to finitary, finite set preserving functors.

▶ **Definition 2.3.** A functor T *preserves finite sets* if $\mathsf{T}X$ is finite whenever X is.

An inclusion preserving functor T is called *finitary* if it satisfies

$$\mathsf{T}X = \bigcup\{\mathsf{T}X' \subseteq \mathsf{T}X \mid X' \subseteq X,\ X' \text{ is finite }\}$$

for all sets X. The finitary version T_ω of an inclusion preserving functor T is defined such that it maps a set X to $\mathsf{T}_\omega X = \bigcup\{\mathsf{T}X' \mid X' \subseteq X,\ X' \text{ is finite }\}$, and a function f to itself.

These definitions can be simply generalized to the class of all set functors.

An example of a finitary version of a functor is $\mathcal{P}_\omega$, that maps a set X to the set of all its finite subsets. An other important class of set functors in the context of coalgebraic modal logic is the class of intersection preserving functors.

▶ **Definition 2.4.** A set functor T *preserves finite intersections* if for all sets A and B, $\mathsf{T}(A \cap B) = \mathsf{T}A \cap \mathsf{T}B$.

2.2 Coalgebras

In the following part of this section, we will briefly recall the basic notions from the theory of coalgebras that we will use later. For a detailed introduction into coalgebras see e.g. [19].

▶ **Definition 2.5.** Given a set functor T, a T-*coalgebra* is a pair $\mathbb{S} = (S, \sigma)$ with $\sigma : S \to \mathsf{T}S$. A pointed T-coalgebra is a pair consisting of a T-coalgebra together with an element of (the carrier set of) that coalgebra. A T-*coalgebra morphism* from T-coalgebra $\mathbb{S} = (S, \sigma)$ to $\mathbb{S}' = (S', \sigma')$, written $f : \mathbb{S} \to \mathbb{S}'$, is a function $f : S \to S'$ such that $\mathsf{T}(f) \circ \sigma = \sigma' \circ f$. The collection of T-coalgebras with their morphisms form a category denoted by $Coalg(\mathsf{T})$.

▶ **Definition 2.6.** Let T be an endofunctor on the category Set, and C an arbitrary set of objects that we shall call *colors*. We let T_C denote the functor $\mathsf{T}_C S = \mathsf{T}S \times C$; that is, T_C maps a set S to the set $\mathsf{T}S \times C$ (and a function $f : S \to S'$ to the function $\mathsf{T}f \times id_C : \mathsf{T}S \times C \to \mathsf{T}S' \times C$). T_C-coalgebras will also be called *C-colored* T-*coalgebras*. We will usually denote T_C-coalgebras as triples $\mathbb{S} = (S, \sigma, \gamma)$, with $\sigma : S \to \mathsf{T}S$ the coalgebra map and $\gamma : S \to C$ the *coloring (marking)*.

▶ **Convention 2.7.** From now on in all our investigations, without lose of generality, we can assume set functor T preserves inclusions and finite intersections. Indeed given any T we can find a naturally isomorphic T' that preserves inclusions and finite intersections. The details can be found in [1]. The important point for us is that the category of $Coalg(\mathsf{T})$ and $Coalg(\mathsf{T}')$ are isomorphic.

2.3 Relation Lifting and Bisimulation

In the remaining part of this section we introduce the notion of relation lifting to define a very general notion of bisimulation for coalgebras. First we recall some central definitions and fix mathematical notation and terminology. Given sets X and Y, we denote a relation R between X and Y by $R : X \nrightarrow Y$ to specify its domain X and codomain Y. We write $R; S : X \nrightarrow Z$ for the composition of two relations $R : X \nrightarrow Y$ and $S : Y \nrightarrow Z$ and $R^\circ : Y \nrightarrow X$ for the converse of $R : X \nrightarrow Y$ with $(y, x) \in R^\circ$ iff $(x, y) \in R$. The graph of any function $f : X \to Y$ is a relation $f : X \nrightarrow Y$ between X and Y for which we also use the symbol f. It will be clear from the context in which a symbol f occurs whether it is meant as a function or a relation. Note that the composition of functions is denoted the other way round than the composition of relations, so we have $g \circ f = f; g$ for functions $f : X \to Y$ and $g : Y \to Z$. For a relation $R : X \nrightarrow Y$ we define the sets

$$Dom(R) = \{x \in X \mid \exists y \in Y, (x, y) \in R\} \subseteq X,$$
$$Rng(R) = \{y \in Y \mid \exists x \in X, (x, y) \in R\} \subseteq Y.$$

The relation $R : X \nrightarrow Y$ is *full* on X if $Dom(R) = X$ and is full on Y if $Rng(R) = Y$. Given sets $X' \subseteq X$ and $Y' \subseteq Y$, we define the restriction $R\mid_{X' \times Y'} : X' \nrightarrow Y'$ of the relation $R : X \nrightarrow Y$ as $R\mid_{X' \times Y'} = R \cap (X \times Y)$. For any set X let $\in_X : X \to \mathcal{P}X$ be the *membership* *relation* between elements of X and subsets of X. Given a set X we define the *diagonal relation* $\Delta_X : X \nrightarrow X$ with $(x, x') \in \Delta_X$ iff $x = x'$. Note that $\Delta_X = id_X$, where id_X is the graph of the identity function.

▶ **Definition 2.8.** A relation lifting L for a set functor T is a collection of relations LR for every relation R, such that $LR : \mathsf{T}X \nrightarrow \mathsf{T}Y$ if $R : X \nrightarrow Y$. We require relation liftings to preserve converse, this means that $L(R^\circ) = (LR)^\circ$ for all relations R.

▶ **Example 2.9.** (i) The *Egli-Milner lifting* $\overline{\mathcal{P}}$ is a relation lifting for covariant power set functor $\mathcal{P}$ that is defined for any $R : X \nrightarrow Y$ such that $\overline{\mathcal{P}}R = \overrightarrow{\mathcal{P}}R \cap \overleftarrow{\mathcal{P}}R$, where:

$$\overrightarrow{\mathcal{P}}R := \{(U, V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall u \in U\ \exists v \in V \text{ s.t. } (u, v) \in R\},$$

$$\overleftarrow{\mathcal{P}}R := \{(U, V) \in \mathcal{P}X \times \mathcal{P}Y \mid \forall v \in V\ \exists u \in U \text{ s.t. } (u, v) \in R\}.$$

(ii) For the constant functor D of a fixed set D define a relation lifting $\overline{D}$ for any $R : X \nrightarrow Y$ such that $\overline{D}R = \Delta_D$.

(iii) Recall the notion of $\overrightarrow{\mathcal{P}}R$ from (i) we can define a relation lifting $\widetilde{\mathcal{M}}$ for the monotone neighborhood functor $\mathcal{M}$ on a relation $R : X \nrightarrow Y$ as follows:

$$\widetilde{\mathcal{M}}R := \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}R \cap \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}R.$$

An important use of relation liftings is to yield a notion of bisimulation.

▶ **Definition 2.10.** Let L be a relation lifting for T and $\mathbb{S} = (S, \sigma)$ and $\mathbb{S}' = (S', \sigma')$ be two T-coalgebras. An *L-bisimulation* between $\mathbb{S}$ and $\mathbb{S}'$ is a relation $R : S \nrightarrow S'$ such that $(\sigma(s), \sigma'(s')) \in LR$, for all $(s, s') \in R$. Two states $s \in \mathbb{S}$ and $s' \in \mathbb{S}'$ are L-bisimilar if there is an L-bisimulation R between $\mathbb{S}$ and $\mathbb{S}'$ with $(s, s') \in R$. We write $\leftrightarroweq^L$ for the notion of L-bisimilarity between fixed coalgebras. Given two C-colored T-coalgebras $\mathbb{S} = (S, \sigma, \gamma)$ and $\mathbb{S}' = (S', \sigma', \gamma')$ and a relation lifting L for T, a relation $R : S \nrightarrow S'$ is an *L_C-bisimulation* between $\mathbb{S}$ and $\mathbb{S}'$, whenever $(\sigma(s), \sigma'(s')) \in LR$ and $\gamma(s) = \gamma'(s')$ for all $(s, s') \in R$.

Now we will give the definition of lax extensions, which are relation liftings satisfying certain conditions that make them well-behaved in the context of coalgebra.

▶ **Definition 2.11.** A relation lifting L for a functor T is called a *lax extension* of T if it satisfies, for all relations $R, R' : X \nrightarrow Z$ and $S : Z \nrightarrow Y$ and all functions $f : X \to Z$:
(L1) $R' \subseteq R$ implies $LR' \subseteq LR$,
(L2) $LR; LS \subseteq L(R; S)$,
(L3) $\mathsf{T}f \subseteq Lf$.
We say that a lax extension L *preserves diagonals* if it additionally satisfies:
(L4) $L\Delta_X \subseteq \Delta_{\mathsf{T}X}$.
We call a lax extension L of T *functorial*, if it distributes over composition, i.e., if $LR; LS = L(R; S)$, and *quasi-functorial*, if

$$LR; LS = L(R; S) \cap (Dom(LR) \times Rng(LS))$$

for all relations $R : X \nrightarrow Z$ and $S : Z \nrightarrow Y$.

▶ **Example 2.12.** The relation lifting $\widetilde{\mathcal{M}}$ which has been defined in Example 2.9, is quasi-functorial.

▶ **Proposition 2.13.** *Let T be a set functor and let L be a quasi-functorial lax extension for T. Then we have:*

(1) *L preserves fullness: If $R : X \nrightarrow Z$ is full on both sides, then so is $LR : \mathsf{T}X \nrightarrow \mathsf{T}Z$;*
(2) *If $R : X \nrightarrow Z$ is full on X and $i : Z \hookrightarrow Z'$ is the inclusion map between Z and Z' then $L(R; i)$ is full on $\mathsf{T}X$;*
(3) *If L preserves diagonals then for any function f, $\mathsf{T}f = Lf$.*

Let us now summarize some facts that we will use about L-bisimulations in the sequel.

▶ **Proposition 2.14.** *For a lax extension L of T and T-coalgebras $\mathbb{S}$, $\mathbb{S}'$ and $\mathbb{Q}$ the following hold:*
(1) *The graph of a coalgebra morphism f from $\mathbb{S}$ to $\mathbb{S}'$ is an L-bisimulation between $\mathbb{S}$ and $\mathbb{S}'$;*
(2) *if $R : S \nrightarrow Q$ respectively $R' : Q \nrightarrow S'$ are L-bisimulations between $\mathbb{S}$ and $\mathbb{Q}$ respectively $\mathbb{Q}$ and $\mathbb{S}'$, then $R; R' : S \nrightarrow S'$ is an L-bisimulation between $\mathbb{S}$ and $\mathbb{S}'$.*

For the proof we refer to [14, Proposition 3].

We will finish this section with a remark on some of the closure properties of the class of functors with a quasi-functorial lax extension:

▶ **Fact 2.15.** *The collection of functors with a quasi-functorial lax extension (FQL) has the following properties:*
1. *the identity functor $I : \mathsf{Set} \to \mathsf{Set}$ is in FQL;*
2. *for each set D, the constant functor $D : \mathsf{Set} \to \mathsf{Set}$ is in FQL;*
3. *the product $X \mapsto \mathsf{T}_1(X) \times \mathsf{T}_2(X)$ of two FQLs T_1 and T_2 is in FQL;*
4. *the coproduct $X \mapsto \mathsf{T}_1(X) + \mathsf{T}_2(X)$ of two FQLs T_1 and T_2 is in FQL;*
5. *the composition $X \mapsto (\mathsf{T}_1 \circ \mathsf{T}_2)(X)$ of a FQL functor T_1 and a functor T_2 which has a functorial lax extension, is in FQL.*

3 Coalgebraic Fixpoint Logic and Automata

3.1 Coalgebraic Fixpoint Logic

In this section we show how to define the syntax and semantics of a coalgebraic fixpoint logic, using a quasi-functorial lax extension L of T. For this purpose from now on we fix a functor T with a quasi-functorial lax extension L. Recall that by our convention 2.7, T preserves all inclusions and finite intersections. We also fix a set P of propositional letters and assume that L preserves diagonals.

▶ **Definition 3.1.** Given a functor T, we define for every set X the function

$$Base : \mathsf{T}_\omega X \to \mathcal{P}_\omega X,\ \alpha \mapsto \bigcap\{X' \subseteq X \mid \alpha \in \mathsf{T}X'\}.$$

The point of this notion is that $Base(\alpha) \in \mathcal{P}_\omega X$ is the least set $U \in \mathcal{P}_\omega X$ such that $\alpha \in \mathsf{T}U$.

The language of the coalgebraic fixpoint logic is defined as follows:

▶ **Definition 3.2.** For P as the set of propositional letters, define the language $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ by the following grammar:

$$a ::= p \mid \neg a \mid \bigvee A \mid \nabla\alpha \mid \mu p.a,$$

where $p \in \mathsf{P}$, $A \in \mathcal{P}_\omega(\mu\mathcal{L}_L^{\mathsf{T}})$ and $\alpha \in \mathsf{T}_\omega(\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P}))$. There is a restriction on the formulation of the formulas $\mu p.a$, namely, no occurrence of p in a may be in the scope of an odd number of negations.[1]

[1] For a precise definition of the notions *scope* and *occurrence*, we can inductively define a construction tree of a formula, where the children of a node labeled $\nabla\alpha$ are given by the formulas in $Base(\alpha)$.

▶ **Remark 3.3.** For a given formula $a \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$, $\mathsf{P}_a \subseteq \mathsf{P}$ denotes the set of all propositional letters occurring in a. Observe that for $\mathsf{Q}' \subseteq \mathsf{Q} \subseteq \mathsf{P}$, we have that $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{Q}') \subseteq \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{Q})$. This can be proved by induction on the complexity of formulas in $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{Q}')$.

Before we turn to the coalgebraic semantics of this language, there are a number of syntactic definitions to be fixed.

▶ **Definition 3.4.** We will write $b \trianglelefteq a$ if b is a subformula of a. Inductively we define the set $\mathit{Sfor}(a)$ of subformulas of a as follows:

$$\begin{array}{rcl} \mathit{Sfor}(p) & := & \{p\}, \\ \mathit{Sfor}(\neg a) & := & \{\neg a\} \cup \mathit{Sfor}(a), \\ \mathit{Sfor}\left(\bigvee A\right) & := & \{\bigvee A\} \cup \bigcup_{a \in A} \mathit{Sfor}(a), \\ \mathit{Sfor}(\mu p.a) & := & \{\mu p.a\} \cup \mathit{Sfor}(a), \\ \mathit{Sfor}(\nabla\alpha) & := & \{\nabla\alpha\} \cup \bigcup_{a \in Base(\alpha)} \mathit{Sfor}(a) \end{array}$$

The elements of $Base(\alpha)$ will be called the immediate subformulas of $\nabla\alpha$.

▶ **Definition 3.5.** A formula $a \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ is *guarded* if every subformula $\mu p.b$ of a has the property that all occurrences of p inside b are within the scope of a ∇.

We now introduce the semantics of coalgebraic fixpoint logic. For this purpose we define the notion of a T-model over a set P of propositional letters.

▶ **Definition 3.6.** A T-model $\mathbb{S} = (S, \sigma, V)$ is a T-coalgebra (S, σ) together with a valuation V that is a function $V : \mathsf{P} \to \mathcal{P}S$.

Using the fixed quasi-functorial lax extension L for the functor T we can define the semantics for the language $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ on T-models, by giving the definition of the satisfaction relation $\Vdash_{\mathbb{S}}: S \nrightarrow \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ for a T-model $\mathbb{S} = (S, \sigma, V)$.

▶ **Definition 3.7.** Before going to the definition of the satisfaction relation, we need to fix some notation: For $X \subseteq S$, $V[p \mapsto X]$ denotes the valuation that is exactly like V apart from mapping p to X. We also use $[\![a]\!]_{\mathbb{S}}$ for the extension of formula a in a T-model $\mathbb{S}$: $[\![a]\!]_{\mathbb{S}} := \{s \in S \mid s \Vdash_{\mathbb{S}} a\}$.Then $[\![a]\!]_{\mathbb{S}[p\mapsto X]}$ denotes the extension of a considering the valuation $V[p \mapsto X]$, instead of V.
Now we are ready to define the satisfaction relation as follows:

$$\begin{array}{rcl} s \Vdash_{\mathbb{S}} p & \text{iff} & s \in V(p) \\ s \Vdash_{\mathbb{S}} \neg a & \text{iff} & \text{not } s \Vdash_{\mathbb{S}} a \\ s \Vdash_{\mathbb{S}} \bigvee A & \text{iff} & s \Vdash_{\mathbb{S}} a \text{ for some } a \in A \\ s \Vdash_{\mathbb{S}} \nabla\alpha & \text{iff} & (\sigma(s), \alpha) \in L \Vdash_{\mathbb{S}} \\ s \Vdash_{\mathbb{S}} \mu p.a & \text{iff} & s \in \bigcap\{X \subseteq S \mid [\![a]\!]_{\mathbb{S}[p\mapsto X]} \subseteq X\}. \end{array}$$

▶ **Remark 3.8.** The clauses in Definition 3.7 are not stated in a correct recursive way. In the recursive clause for the ∇ modality we make use of the unrestricted satisfaction relation $\Vdash_{\mathbb{S}}$ that has yet to be defined. We can only suppose that $\Vdash_{\mathbb{S}}|_{S\times Base(\alpha)}$ is already defined. The actual recursive definition is that $s \Vdash_{\mathbb{S}} \nabla\alpha$ iff $(\sigma(s), \alpha) \in L(\Vdash_{\mathbb{S}}|_{S\times Base(\alpha)})$. To see why this is equal to the clause given above, see [14, Proposition 6].

▶ **Remark 3.9.** Given a valuation $V : \mathsf{P} \to \mathcal{P}S$, one can think of it as a coloring $\gamma_V : S \to \mathcal{P}\mathsf{P}$ given by: $\gamma_V(s) := \{p \in \mathsf{P} \mid s \in V(p)\}$. So following Definition 2.6, a T-model $\mathbb{S} = (S, \sigma, V)$ can also be seen as a $\mathcal{P}(\mathsf{P})$-colored T-coalgebra denoted as $\hat{\mathbb{S}} = (S, \sigma, \gamma_V)$.

▶ **Definition 3.10.** The *projection* of a $\mathcal{P}(\mathsf{P})$-colored T-coalgebra $\mathbb{S} = (S, \sigma, \gamma)$ to a set $\mathsf{Q} \subseteq \mathsf{P}$ is the $\mathcal{P}(\mathsf{Q})$-colored T-coalgebra $\mathbb{S}^{\mathsf{Q}} = (S, \sigma, \gamma^{\mathsf{Q}})$ where $\gamma^{\mathsf{Q}} : S \to \mathcal{P}\mathsf{Q}$, $s \mapsto \gamma(s) \cap \mathsf{Q}$.

▶ **Definition 3.11.** Given a set $\mathsf{Q} \subseteq \mathsf{P}$, an L_{Q}-*bisimulation* between two T-models $\mathbb{S}$ and $\mathbb{Y}$ is defined to be an $L_{\mathcal{P}(\mathsf{Q})}$-bisimulation between $\mathcal{P}(\mathsf{Q})$-colored T-colagebras $\hat{\mathbb{S}}^{\mathsf{Q}}$ and $\hat{\mathbb{Y}}^{\mathsf{Q}}$, which are given by Remark 3.9 and Definition 3.10. More precisely, a relation $R : S \not\to Y$ is an L_{Q}-bisimulation between T-models $\mathbb{S} = (S, \sigma, V_S)$ and $\mathbb{Y} = (Y, \lambda, V_Y)$ if and only if R is an L-bisimulation between T-coalgebras $\mathbb{S} = (S, \sigma)$ and $\mathbb{Y} = (Y, \lambda)$ and R preserves the truth of all propositional letters in Q, that is, for all $(s, y) \in R$ and $p \in \mathsf{Q}$, $s \in V_S(p)$ iff $y \in V_Y(p)$.

From this definition, it is easy to see that for any $\mathsf{Q}' \subseteq \mathsf{Q}$, if a relation R is an L_{Q}-bisimulation between T-models $\mathbb{S}$ and $\mathbb{Y}$, then it is also an $L_{\mathsf{Q}'}$-bisimulation between them.

▶ **Definition 3.12.** Given a propositional letter $p \in \mathsf{P}$, a relation $R : S \not\to S'$ is an *up-to-p L_{P}-bisimulation* between two T-models $\mathbb{S} = (S, \sigma, V)$ and $\mathbb{S}' = (S', \sigma', V')$, if it is an $L_{\mathsf{P}\setminus\{p\}}$-bisimulation between T-models $\mathbb{S}$ and $\mathbb{S}'$. We write $s \leftrightarrows_p^L s'$ if s and s' are up-to-p L_{P}-bisimilar, that is where we disregard the proposition letter p.

Now we are going to look at the expressive power of $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ with respect to states in T-models. For this, we start with a definition.

▶ **Definition 3.13.** Two states s in T-model $\mathbb{S} = (S, \sigma, V)$ and s' in T-model $\mathbb{S}' = (S', \sigma', V')$ are called *equivalent* for formulas in $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ if $s \Vdash_{\mathbb{S}} a$ iff $s' \Vdash_{\mathbb{S}'} a$, for all $a \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$.

An important property of our coalgebraic fixpoint logic is that truth is bisimulation invariant. This fact is given by the following proposition.

▶ **Proposition 3.14.** *Given a state s in a T-model $\mathbb{S} = (S, \sigma, V)$ and a state s' in a T-model $\mathbb{S}' = (S', \sigma', V')$, if s and s' are L_{P}-bisimilar then s and s' are equivalent for formulas in $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$.*

For the proof of this proposition we refer to [22, Proposition 5.14], [13, Proposition 4.11] and the fact that lax extensions are monotone.

Now we are ready to state the last semantic result we will need throughout this paper.

▶ **Proposition 3.15.** *Each formula in $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ can be transformed into an equivalent guarded formula in $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$.*

It can be proved by induction on the complexity of formulas, see [22, Proposition 5.15]

▶ **Convention 3.16.** Throughout this paper we always assume $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$-formulas to be guarded.

3.2 Coalgebraic Automata

Coalgebraic automata are supposed to operate on pointed coalgebras. Basically, the idea is that an initialized T-automaton will either *accept* or *reject* a given pointed T-coalgebra. In the following section, we will recall the basic definitions from coalgebraic automata theory.

▶ **Definition 3.17.** Given a functor $\mathsf{T} : \mathsf{Set} \to \mathsf{Set}$. A (non-deterministic) T-automaton over a color set C is a triple $\mathbb{A} = (A, \Delta, \Omega)$, with A some finite set (of states), $\Delta : A \times C \to \mathcal{P}\mathsf{T}A$ the *transition map* and $\Omega : A \to \omega$ a *parity map*. An *initialized* version of $\mathbb{A}$ is a pair $(\mathbb{A}, a)$ consisting of an automaton $\mathbb{A}$ together with an element $a \in A$, which is the *initial* state.

The acceptance condition for T-automaton is formulated in terms of a parity game[12]. The acceptance game $\mathcal{G}(\mathbb{S}, \mathbb{A})$ between initialized automaton $(\mathbb{A}, a_I)$ and a pointed coalgebra $(\mathbb{S}, s_I)$ is given by the Table 1. The game is played by two players: Éloise ($\exists$) and Abélard ($\forall$). A *match* of the game is a (finite or infinite) sequence of positions which is given by the two players moving from one position to another according to the rules of Table 1. Let us now give the formal definition of acceptance game.

▶ **Definition 3.18.** Let $(\mathbb{A}, a_I)$ be an initialized T-automaton over the color set C. Furthermore let $(\mathbb{S}, s_I) = (S, \sigma, \gamma, s_I)$ be a pointed C-colored T-coalgebra. Then the *acceptance game* $\mathcal{G}(\mathbb{S}, \mathbb{A})$ is given by the following table:

Position	Player	Admissible moves	Parity
$(s, a) \in S \times A$	$\exists$	$(\sigma(s), \phi)$ *s.t.* $\phi \in \Delta(a, \gamma(s))$	$\Omega(a)$
$(\sigma(s), \phi) \in \mathsf{T}S \times \mathsf{T}A$	$\exists$	$\{Z : S \not\rightarrow A \mid (\sigma(s), \phi) \in LZ$	0
$Z \subseteq S \times A$	$\forall$	Z	0

Table 1 Acceptance game for T-automaton

Positions of the form $(s, a) \in S \times A$ will be called *basic positions* of the game. A partial play of the game of the form $(s, a)(\sigma(s), \phi)Z(t, b)$ (with $(s, a) \in S \times A$, $(\sigma(s), \phi) \in \mathsf{T}S \times \mathsf{T}A$, $Z : S \not\rightarrow A$ and $(t, b) \in Z$) will be called a *round* of the play. For the winning conditions, recall that finite matches are lost by the player who gets stuck. For infinite matches, consider an arbitrary such match:

$$\rho = (s_0, a_0)(\sigma(s_0), \phi_0)Z_0(s_1, a_1)(\sigma(s_1), \phi_1)Z_1(s_2, a_2)...$$

Clearly, ρ induces an infinite sequence of basic positions $(s_0, a_0)(s_1, a_1)(s_2, a_2)...$ and, thus, an infinite sequence of states in A: $\rho \restriction_A := a_0 a_1 a_2...$ Now $\exists$ is the winner of the match ρ if the maximum priority occurring infinitely often on $\rho \mid_A$ is even. Otherwise $\forall$ wins ρ. A *positional* or *history free strategy* for $\exists$ is a pair of functions $(\Phi : S \times A \to \mathsf{T}A, Z : S \times A \to \mathcal{P}(S \times A))$. Such a strategy is *legitimate* if at any position, it maps the position to an admissible next position. A legitimate strategy is *winning* for $\exists$ from a position in the game, if it guarantees $\exists$ to win any match starting from that position, no matter how $\forall$ plays. A position starting from which $\exists$ has a winning strategy is called a *winning position* for $\exists$. The set of all winning positions for $\exists$ in $\mathcal{G}(\mathbb{S}, \mathbb{A})$ is denoted by $\mathrm{Win}_\exists(\mathbb{S}, \mathbb{A})$ or shortly by $\mathrm{Win}_\exists$. A history-free strategy (Φ, Z) initialized at $(s_I, b) \in S \times A$ is called *scattered* if the relation

$$\{(s_I, b)\} \cup \bigcup \{Z_{s,a} \subseteq S \times A \mid (s, a) \in \mathrm{Win}_\exists\},$$

with $Z_{s,a}$ the value of Z on (s, a), is functional. Finally we say that initialized T-automaton $(\mathbb{A}, a_I)$ *accepts* $(\mathbb{S}, s_I)$ if $\exists$ has a winning strategy in the game $\mathcal{G}(\mathbb{A}, \mathbb{S})$ initialized at position (s_I, a_I). If $\exists$ has a scattered winning strategy starting from (s_I, a_I), we will say $(\mathbb{A}, a_I)$ *strongly accepts* $(\mathbb{S}, s_I)$.

▶ **Definition 3.19.** For every initialized T-automaton $(\mathbb{A}, a_I)$ over some color set C, $L(\mathbb{A}, a_I)$, the *recognizable language* of $(\mathbb{A}, a_I)$, is the class of all pointed C-colored T-coalgebras that are accepted by $(\mathbb{A}, a_I)$. We call two initialized T-automata $(\mathbb{A}, a_I)$ and $(\mathbb{A}', a'_I)$ over set C *equivalent* iff $L(\mathbb{A}, a_I) = L(\mathbb{A}', a'_I)$.

3.3 Logic and Automata

There is a routine construction of an equivalent initialized T-automaton $(\mathbb{A}, a_I)$ from a $\mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$-formula, and vice versa. Given the finitary nature of our automata, this construction requires the functor T to preserve finite sets.

▶ **Proposition 3.20.** *Let T be a functor that preserves finite sets. There exists an effective procedure to transform a formula $b \in \mu\mathcal{L}_L^{T}(P)$ to an initialized T-automaton $(\mathbb{A}_b, a_b)$ over the set $C = \mathcal{P}(P)$ such that for every C-colored T-coalgebra $(\mathbb{S}, s)$: $(\mathbb{S}, s) \Vdash_{\mathbb{S}} b$ iff $(\mathbb{A}_b, a_b)$ accepts $(\mathbb{S}, s)$.*

Proof sketch. Our construction proceeds along the exact same lines as the construction of an initialized *alternating* T-automaton from a given formula [22, Theorem 2] and transforming it to a non-deterministic T-automaton [12, Theorem 1] in the case of a functor that preserves weak pullbacks, and uses some facts from [13] to the fact that also in our case, the nabla operator has certain desirable properties. The construction proceeds in the following stages:

(0) First of all we generalize our notion of a T-automaton (which is non-deterministic in nature) to that of an alternating T-automaton, which has a transition map of the type $\Delta : A \to \mathcal{P}\mathcal{P}\mathsf{T}A$. For these automata, acceptance is defined as for the alternating automata in [12] and [22].

(1) Using routine methods [22, Theorem 2] we can inductively show that every formula in our language can be effectively transformed into an equivalent alternating automaton. For the case of negation we use the method of [11] together with the fact that the dual of our nabla operator can be expressed using disjunctions and the nabla operator itself [13, Theorem 4.14].

(2) What is still missing is a *simulation theorem* stating that every alternating automaton can be replaced with an equivalent non-deterministic one. This result is in fact also a more or less routine result [2, section 9.6], since we can use the fact that our nabla also satisfies a certain modal distributive law, stating that the conjunction of nablas over some formulas is equivalent to a disjunction of nablas over some conjunctions of these formulas [13, Proposition 4.17].

(3) Combining (1) and (2) we see that every formula in our coalgebraic fixpoint language is equivalent to one of our automata indeed.

◀

▶ **Proposition 3.21.** *There exists an effective procedure transforming an initialized T-automaton $(\mathbb{A}, a_I)$ to an equivalent $\mu\mathcal{L}_L^{T}(P)$-formula $a_{\mathbb{A}}$.*

This result is rather standard, see for instance [22, Theorem 3].

4 Automata are Closed under Projection

This section is devoted to the proof of our main technical result i.e., closure under projection.

▶ **Definition 4.1.** Let $\mathbb{A} = (A, \Delta, \Omega)$ be a T-automaton over color set C. We call a state $a \in A$ a *true state* of $\mathbb{A}$ if $\Omega(a)$ is even and $\Delta(a, c) = \mathsf{T}(\{a\})$. We will standardly use the notation $a_\top$ to refer to a true state. Given $(a, c) \in A \times C$ we call $\phi \in \Delta(a, c)$ a *satisfiable element* of $\mathbb{A}$ if there is a witnessing T-coalgebra $(\mathbb{Q}_\phi, \rho, \gamma_Q)$, $\tau \in \mathsf{T}Q$ and a relation $Z_\phi : Q \nrightarrow A$ such that $(\tau, \phi) \in LZ_\phi$ and $Z_\phi \subseteq \mathrm{Win}_\exists(\mathbb{Q}, \mathbb{A})$. Finally we call a T-automaton $\mathbb{A}$ *totally satisfiable* whenever for all $(a, c) \in A \times C$ and $\phi \in \Delta(a, c)$, ϕ is satisfiable.

The following proposition states that without loss of generality we can always assume that an initialized T-automaton $(\mathbb{A}, a_I)$ is totally satisfiable and has a true state. Furthermore, we may always assume that there exists a witnessing T-coalgebra $\mathbb{Q}$ that works for all $(a, c) \in A \times C$ and $\phi \in \Delta(a, c)$.

▶ **Proposition 4.2.** *For any initialized T-automaton $(\mathbb{A}, a_I)$ over a color set C we have that:*

1. *There is an equivalent initialized T-automaton $(\mathbb{A}', a_I)$ such that $\mathbb{A}'$ has a true state.*
2. *There exists a totally satisfiable initialized T-automaton $(\mathbb{A}', a_I')$ which is equivalent to $(\mathbb{A}, a_I)$.*
3. *If $(\mathbb{A}, a_I)$ is totally satisfiable, then there is a C-colored witnessing coalgebra $\mathbb{Q} = (Q, \rho, \gamma_Q)$ and a relation $Y : Q \nrightarrow A$ with $Y \subseteq Win_\exists(\mathbb{Q}, \mathbb{A})$ such that for all $(a, c) \in A \times C$ and $\phi \in \Delta(a, c)$, there is a $\tau \in \mathsf{T}Q$ such that $(\tau, \phi) \in LY$.*

Now we will state the main technical result of this paper. Theorem 4.3 is a generalization of [12, Proposition 5.9], where the same result is proved for the weak-pullback preserving functors. In the following theorem we will generalize the proposition to the class of all functors with a quasi-functorial lax extension that preserves diagonals. The proof strategy is the same as in [12], but the construction here is more involved.

▶ **Theorem 4.3 (Closure under projection).** *Given an initialized T-automaton $(\mathbb{A}, a_I)$ over a color set $\mathcal{P}(\mathsf{P})$ and an element $p \in \mathsf{P}$, then there exists an initialized T-automaton $(\exists_p.\mathbb{A}, a_I)$ over the color set $\mathcal{P}(\mathsf{P} \setminus \{p\})$ such that:*

$$(\mathbb{S}, s_I) \in L(\exists_p.\mathbb{A}, a_I) \textit{ iff } (\overline{\mathbb{S}}, \overline{s}_I) \in L(\mathbb{A}, a_I) \textit{ for some } (\overline{\mathbb{S}}, \overline{s}_I) \textit{ with } \mathbb{S}, s_I \leftrightarrows_p^L \overline{\mathbb{S}}, \overline{s}_I. \tag{2}$$

Proof. Given $(\mathbb{A}, a_I)$ over color a set $\mathcal{P}\mathsf{P}$, we define the initialized T-automaton $(\exists_p.\mathbb{A}, a_I)$ over the color set $\mathcal{P}(\mathsf{P} \setminus \{p\})$ as the automaton $(\exists_p.\mathbb{A}, a_I) := (A, \Delta_p, \Omega, a_I)$, where

$$\Delta_p : A \times \mathcal{P}(\mathsf{P} \setminus \{p\}) \to \mathcal{P}\mathsf{T}A, \ (a, c) \mapsto \Delta(a, c) \cup \Delta(a, c \cup \{p\}).$$

We need to show that $(\exists_p.\mathbb{A}, a_I)$ satisfies (2). The right-to-left direction of (2) is straightforward, since all legitimate moves of $\exists$ in the game $\mathcal{G}(\mathbb{A}, \mathbb{S}')$ are also legitimate in $\mathcal{G}(\exists_p.\mathbb{A}, {\mathbb{S}'}_p)$.

To show the left-to-right direction of (2) assume that $(\exists_p.\mathbb{A}, a_I)$ accepts the $\mathcal{P}(\mathsf{P} \setminus \{p\})$-colored T-coalgebra $(\mathbb{S}, s_I) = (S, \sigma, \gamma, s_I)$. We need to define a $\mathcal{P}\mathsf{P}$-colored coalgebra $(\overline{\mathbb{S}}, \overline{s}_I) = (\overline{S}, \overline{\sigma}, \overline{\gamma}, \overline{s}_I) \in L(\mathbb{A}, a)$ that is up-to-p bisimilar to $(\mathbb{S}, s_I)$.

By Proposition 4.2 we can assume that $(\exists_p.\mathbb{A}, a_I)$ has a true state and is totally satisfiable, which entails that there is a $\mathcal{P}\mathsf{P}$-colored coalgebra $\mathbb{Q} = (Q, \rho, \gamma_Q)$ and a relation $Y : Q \nrightarrow A$ with $Y \subseteq \mathrm{Win}_\exists(\mathbb{Q}, \exists_p.\mathbb{A})$ such that for all $(a, c) \in A \times C$ and $\phi \in \Delta(a, c)$ there is a $\tau \in \mathsf{T}Q$ with $(\tau, \phi) \in LY$.

The carrier of $(\overline{\mathbb{S}}, \overline{s}_I)$ is the set $\overline{S} := (S \times A) \uplus Q$. To define the coalgebra structure $\overline{\sigma} : \overline{S} \to \mathsf{T}\overline{S}$ we distinguish the following cases:

(1) If $q \in Q$, define $\overline{\sigma}(q) := \rho(q)$.

(2) If $(s, a) \in S \times A$ and $(s, a) \notin \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$, define $\overline{\sigma}(s, a) := \mathsf{T}\kappa_a(\sigma(s))$, where $\kappa_a : S \to S \times A$, $s \mapsto (s, a)$.

(3) In the case where $(s, a) \in S \times A$ and $(s, a) \in \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ we define $\overline{\sigma}(s, a)$ as follows: From the winning strategy that witnesses $(s, a) \in \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ we obtain a $\phi_{s,a} \in \Delta(a, \gamma(s))$ and a relation $Z_{s,a} : S \nrightarrow A$ such that $Z_{s,a} \subseteq \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ and $(\sigma(s), \phi_{s,a}) \in LZ_{s,a}$. Because $(\mathbb{A}, a_I)$ contains a true state we can assume without loss of generality that $Z_{s,a}$ is full on S. We can write $Z_{s,a} = \pi_1^\circ ; \pi_2$ where $\pi_1 : Z_{s,a} \to S$ and $\pi_2 : Z_{s,a} \to A$ are the projections of $Z_{s,a}$. These projections can be seen as relations with domain $(S \times A) \uplus Q$ for which it then follows that $Z_{s,a} \subseteq \pi_1^\circ ; (\pi_s \uplus Y)$. Because L is a lax extension

one obtains that $LZ_{s,a} \subseteq L(\pi_1^\circ; (\pi_s \uplus Y))$ and hence $(\sigma(s), \phi_{s,a}) \in L(\pi_1^\circ; (\pi_s \uplus Y))$. It also holds that $\sigma(s) \in Dom(L(\pi_1^\circ))$ because $Z_{s,a}$ is full on S, so π_1° is full on S, and hence by Proposition 2.13 (2) $L(\pi_1^\circ)$ is full on $\mathsf{T}S$. Moreover $\phi_{s,a} \in Rng(L(\pi_2 \uplus Y))$ because $\phi_{s,a} \in Rng(LY)$ by the properties of Y and $LY \subseteq L(L(\pi_2 \uplus Y)$.

With the quasi-functoriality of L it now follows that $(\sigma(s), \phi_{s,a}) \in L(\pi_1^\circ); L(\pi_s \uplus Y)$. Hence it is possible to choose $\overline{\sigma}(s,a) \in \mathsf{T}((S \times A) \uplus Q)$ such that

$$(\sigma(s), \overline{\sigma}(s,a)) \in L(\pi_1^\circ) \text{ and } (\overline{\sigma}(s,a), \phi_{s,a}) \in L(\pi_2 \uplus Y).$$

To complete the definition of the $\mathcal{P}\mathsf{P}$-colored pointed coalgebra $(\overline{\mathbb{S}}, \overline{s_I})$ we set $\overline{s_I} := (s_I, a_I)$ and define the coloring $\overline{\gamma} : \overline{S} \to \mathcal{P}(\mathsf{P})$ by distinguishing the following cases:

(1) If $q \in Q$, define $\overline{\gamma}(q) := \gamma_Q(q)$.
(2) If $(s,a) \in S \times A$ and $(s,a) \notin \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$, define $\overline{\gamma}(s,a) := \gamma(s)$.
(3) If $(s,a) \in S \times A$ and $(s,a) \in \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ we define $\overline{\gamma}(s,a)$ by considering the choice of $\exists$ at (s,a). Since (s,a) is a winning position for $\exists$, she picks an element $\phi_{s,a} \in \Delta_p(a, \gamma(s))$. The function Δ_p is defined such that $\Delta_p(a, \gamma(s)) = \Delta(a, \gamma(s)) \cup \Delta(a, \gamma(s) \cup \{p\})$. We set

$$\overline{\gamma}(s,a) := \begin{cases} \gamma(s) \cup \{p\} & \text{if } \phi_{s,a} \in \Delta(a, \gamma(s) \cup \{p\}), \\ \gamma(s) & \text{otherwise.} \end{cases}$$

We need to show that $\mathbb{S}, s_I \leftrightarrow_p^L \overline{\mathbb{S}}, (s_I, a_I)$ and that $((s_I, a_I), a_I) \in \mathrm{Win}_\exists(\overline{\mathbb{S}}, \mathbb{A})$.

▶ **Claim (1).** $\mathbb{S}, s_I \leftrightarrow_p^L \overline{\mathbb{S}}, (s_I, a_I)$.

Proof of claim (1): We show that graph of the projection $\pi_S : S \times A \to S$ seen as a relation between $\overline{S}$ and S is an up-to-p bisimulation between $\overline{\mathbb{S}}, \overline{s_I}$ and $\mathbb{S}, s_I$. We need to prove that

$$(\overline{\sigma}(s,a), \sigma(s)) \in L\pi_S \text{ and } \overline{\gamma}(s,a) \setminus \{p\} = \gamma(s) \text{ whenever } ((s,a),s) \in \pi_1.$$

That $\overline{\gamma}(s,a) \setminus \{p\} = \gamma(s)$ follows directly from the definition of $\overline{\gamma}$. For $(\overline{\sigma}(s,a), \sigma(s)) \in L\pi_S$ we distinguish two cases:

(i) If $(s,a) \in S \times A$ and $(s,a) \notin \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ then the statement holds because by definition $\overline{\sigma}(s,a) = \mathsf{T}\kappa_a(\sigma(s))$ and since L is a lax extensions and $\kappa_a \subseteq \pi_S$ we have that

$$(\mathsf{T}\kappa_a(\sigma(s)), \sigma(s)) \in \mathsf{T}\kappa_a = L\kappa_a \subseteq L\pi_S$$

(ii) If $(s,a) \in S \times A$ and $(s,a) \in \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$ then we get by the definition of $\overline{\sigma}$ that $(\overline{\sigma}(s,a), \sigma(s)) \in L\pi_1$. It follows that $(\overline{\sigma}(s,a), \sigma(s)) \in L\pi_S$ because L is a lax extensions and $\pi_1 \subseteq \pi_S$ since $\pi_1 : Z_{s,a} \to S$ is the projection of the relation $Z_{s,a} \subseteq S \times A$.

▶ **Claim (2).** $((s_I, a_I), a_I) \in \mathrm{Win}_\exists(\overline{\mathbb{S}}, \mathbb{A})$.

Proof of claim (2): Let (Ψ, Y') be a strategy for $\exists$ witnessing that $Y \subseteq \mathrm{Win}_\exists(\mathbb{Q}, \exists_p.\mathbb{A})$. Define $\exists$'s strategy in $\mathcal{G}(\overline{\mathbb{S}}, \mathbb{A})$ as follows:

$$\begin{aligned} \overline{\Phi} : \overline{S} \times A &\to \mathsf{T}A & \overline{Z} : \overline{S} \times A &\to \mathcal{P}(\overline{S} \times A) \\ ((s,b),a) &\mapsto \phi_{s,a} & ((s,b),a) &\mapsto \pi_2 \uplus Y \\ (q,a) &\mapsto \psi_{q,a} & (q,a) &\mapsto Y'_{q,a} \end{aligned}$$

where π_2 is the projection of $Z_{s,a}$, if $(s,a) \in \mathrm{Win}_\exists(\mathbb{S}, \exists_p.\mathbb{A})$, and arbitrary otherwise.

▶ **Claim (2a).** For the following types of positions in $\mathcal{G}(\overline{\mathbb{S}}, \mathbb{A})$, the given strategy $(\overline{\Phi}, \overline{Z})$ provides legitimate moves for $\exists$:

(i) $(q,a) \in \overline{S} \times A$ and $(q,a) \in \mathrm{Win}_\exists(\mathbb{Q}, \exists p.\mathbb{A})$,

(ii) $((s,a),a) \in \overline{S} \times A$ and $(s,a) \in \mathrm{Win}_{\exists}(\mathbb{S}, \exists_p.\mathbb{A})$

Proof of Claim (2a):
(i) This is clear since $\overline{\sigma}(q) = \rho(q)$ and $\exists$ plays her winning strategy in $\mathcal{G}(\mathbb{Q}, \exists p.\mathbb{A})$.
(ii) By the definition of $\overline{\gamma}$ we have that $\overline{\phi}_{s,a} = \phi_{s,a} \in \Delta(a, \overline{\gamma}(s,a))$. Also $(\overline{\sigma}(s,a), \overline{\phi}_{s,a}) \in L\overline{Z}_{s,a}$ because $(\overline{\sigma}(s,a), \phi_{s,a}) \in L(\pi_2 \uplus Y)$.

▶ **Claim (2b).** $(\overline{\Phi}, \overline{Z})$ guarantees $\exists$ to win any match of $\mathcal{G}(\overline{\mathbb{S}}, \mathbb{A})$ starting from $((s_I, a_I), a_I)$.

Proof of Claim (2b): To check that $(\overline{\Phi}, \overline{Z})$ is winning it suffices to distinguish the following two kinds of matches:
(i) At some stage $\forall$ chooses an element $(q,a) \in Y$. From this moment on, there is no way to go through the states of $\mathbb{S}$ and since $Y \subseteq \mathrm{Win}_{\exists}(\mathbb{Q}, \exists_p.\mathbb{A})$, $\exists$ plays her winning strategy in $\mathcal{G}(\mathbb{Q}, \mathbb{A})$ and wins the match.
(ii) $\forall$ never picks an element of the form (q,a). In this case any $(\overline{\Phi}, \overline{Z})$-conforming match is of the form

$$((s_I, a_I), a_I)((s_1, a_1), a_1)((s_2, a_2), a_2)\dots$$

This match corresponds to the (Φ, Z)-conforming match

$$(s_I, a_I)(s_1, a_1)(s_2, a_2)\dots$$

in the game $\mathcal{G}(\mathbb{S}, \mathbb{A})$. Since we assumed (Φ, Z) to be a winning strategy for $\exists$, $(\overline{\Phi}, \overline{Z})$ is also a winning strategy for her.

This finishes the proof of Theorem 4.3. ◀

5 Uniform Interpolation for $\mu\mathcal{L}_L^{\mathsf{T}}$

In the following section we will prove the main theorem of this paper, viz., uniform interpolation for $\mu\mathcal{L}_L^{\mathsf{T}}$. Our proof follows and generalizes the proof in [20] which shows a similar result for monotone modal logic (without fixpoints). We first need some auxiliary definitions.

▶ **Definition 5.1.** Define the relation of logical consequence $\models: \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P}) \nrightarrow \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ by $a \models a'$ if and only if $s \Vdash_{\mathbb{S}} a$ implies $s \Vdash_{\mathbb{S}} a'$ for all states s in any T-model $\mathbb{S}$.

▶ **Definition 5.2.** Given a formula $a \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$, we let P_a denote the (obviously defined) set of proposition letters occurring in a.

Our main result can now be formulated as follows.

▶ **Theorem 5.3** (Uniform Interpolation). *Let T be a set functor that preserves finite sets, and let L be a quasi-functorial lax extension for T. For any formula $a \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ and any set $\mathsf{Q} \subseteq \mathsf{P}_a$ of propositional letters, there is a formula $a_{\mathsf{Q}} \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{Q})$, effectively constructable from a, such that for every formula $b \in \mu\mathcal{L}_L^{\mathsf{T}}(\mathsf{P})$ with $\mathsf{P}_a \cap \mathsf{P}_b \subseteq \mathsf{Q}$, we have that*

$$a \models b \quad \textit{iff} \quad a_{\mathsf{Q}} \models b.$$

If a is fixpoint-free, then so is a_{Q}.

As mentioned in the introduction, our proof is based on the definability of the bisimulation quantifier in our language.

▶ **Proposition 5.4.** *Given any proposition letter p, there is a map $\exists p : \mu\mathcal{L}^{\mathsf{T}}_L(\mathsf{P}) \longrightarrow \mu\mathcal{L}^{\mathsf{T}}_L(\mathsf{P})$ such that $\mathsf{P}_{\exists p.b} = \mathsf{P}_b \setminus \{p\}$ and*

$$\mathbb{S}, s \Vdash \exists p.b \quad \textit{iff} \quad \mathbb{S}', s' \Vdash b, \ \textit{for some } \mathbb{S}', s' \textit{ with } \mathbb{S}, s \leftrightarrows^L_p \mathbb{S}', s'. \tag{3}$$

for any formula $b \in \mu\mathcal{L}^{\mathsf{T}}_L(\mathsf{P})$.

Proof. Take a formula $b \in \mu\mathcal{L}^{\mathsf{T}}_L(\mathsf{P})$. By Proposition 3.20 we can transform it to an equivalent initialized T-automaton $(\mathbb{A}_b, a_b)$. From Theorem 4.3 we have an initialized T-automaton $(\exists_p.\mathbb{A}_b, a_b)$ such that:

$(\exists_p.\mathbb{A}_b, a_b)$ accepts $(\mathbb{S}, s)$ iff $(\mathbb{A}_b, a_b)$ accepts $(\mathbb{S}', s')$ for some $(\mathbb{S}', s')$ with $\mathbb{S}, s \leftrightarrows^L_p \mathbb{S}', s'$.

Now by Proposition 3.21 we can transform the initialized T-automaton $(\exists_p.\mathbb{A}_b, a_b)$ to an equivalent formula $a_{(\exists_p.\mathbb{A}_b)}$ and put $\exists p.b := a_{(\exists_p.\mathbb{A}_b)}$. It is easy to show that:

$$\mathbb{S}, s \Vdash a_{(\exists_p.\mathbb{A}_b)} \quad \text{iff} \quad \mathbb{S}', s' \Vdash b, \ \text{for some } \mathbb{S}', s' \text{ with } \mathbb{S}, s \leftrightarrows^L_p \mathbb{S}', s'.$$

We leave it for the reader to verify that $\mathsf{P}_{\exists p.b} = \mathsf{P}_b \setminus \{p\}$. ◀

Now we are ready to prove the uniform interpolation theorem:

Proof of Theorem 5.3 Let $p_0, p_1, ..., p_{n-1}$ enumerate the proposition letters in $\mathsf{P}_a \setminus \mathsf{Q}$, and set

$$a_{\mathsf{Q}} := \exists p_0 \exists p_1 \dots \exists p_{n-1}.a.$$

It is not difficult to verify that a_{Q} is fixpoint-free if a is so.

In order to check that $a \vDash b$ iff $a_{\mathsf{Q}} \vDash b$, first assume that $a \vDash b$. To prove that $a_{\mathsf{Q}} \vDash b$ take a pointed T-model $(\mathbb{S}_0, s_0)$ with $s_0 \Vdash_{\mathbb{S}_0} a_{\mathsf{Q}}$. By the semantics of the bisimulation quantifiers we get states s_i in T-models $\mathbb{S}_i$ for $i = 1, 2, \dots, n$ such that $s_i \leftrightarrows_{p_i} s_{i+1}$ for $i = 0, ..., n$ and $s_n \Vdash_{\mathbb{S}_n} a$. From the latter fact it follows that $s_n \Vdash_{\mathbb{S}_n} b$ since we have assumed $a \vDash b$. Because each of the witnessing up-to-p_i L_{P}-bisimulations for $i = 0, 1, \dots, n-1$ is also an $L_{\mathsf{P} \setminus \{p_0, p_1, \dots, p_{n-1}\}}$-bisimulation, we can compose them and obtain an $L_{\mathsf{P} \setminus \{p_0, p_1, \dots, p_{n-1}\}}$-bisimulation between s_0 and s_n. Since $\mathsf{P}_b \subseteq \mathsf{P} \setminus \{p_0, p_1, \dots, p_{n-1}\}$ we get $s_0 \Vdash_{\mathbb{S}_0} b$.

For the other direction, we show that $a \vDash a_{\mathsf{Q}}$, then $a \vDash b$ follows by transitivity from $a_{\mathsf{Q}} \vDash b$. Take any state s in a T-model $\mathbb{S} = (S, \sigma, V)$ with $s \Vdash_{\mathbb{S}} a$. Then $s \Vdash_{\mathbb{S}} a_{\mathsf{Q}}$ because s is up-to-p L_{P}-bisimilar to itself for any $p \in \mathsf{P}$, since the identity on S is an L_{P}-bisimulation. ◀

6 Conclusions and Future Work

In this paper we showed that the coalgebraic fixpoint logic for functors with a quasi-functorial lax extension that preserves diagonals, enjoys uniform interpolation. This suggests to further study the class of functors possessing such a relation lifting. For instance one might try to characterize this class of functors in categorical terms and investigate how the cover modality of such a relation lifting relates to modalities arising from predicate liftings [14].

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Appendix

In this appendix we gather some proofs that we had to omit from the main text for reasons of space limitations.

Proof of Proposition 2.2

Let $f : A \to B$ be a set function and $f|_{Rng(f)} : A \to Rng(f)$ denote the restriction of f on its range. So $f = i \circ f|_{Rng(f)}$, where $i : Rng(f) \hookrightarrow B$ is the inclusion map. Hence $\mathsf{T}(f) = \mathsf{T}(i \circ f|_{Rng(f)}) = \mathsf{T}i \circ \mathsf{T}(f|_{Rng(f)})$ and $Rng(\mathsf{T}f) = Rng(\mathsf{T}i \circ \mathsf{T}(f|_{Rng(f)}))$. But the inclusion preserving property of T implies that $\mathsf{T}i$ is the inclusion map from $\mathsf{T}(Rng(f))$ to $\mathsf{T}B$, and from this fact it follows that $Rng(\mathsf{T}i \circ \mathsf{T}(f|_{Rng(f)})) = Rng(\mathsf{T}(f|_{Rng(f)}))$. But since $f|_{Rng(f)}$ is a surjective map and set functors preserve the surjectiveness of functions, we have $Rng(\mathsf{T}(f|_{Rng(f)})) = \mathsf{T}(Rng(f))$. Hence $\mathsf{T}(Rng(f)) = Rng\mathsf{T}(f)$.

Proof of quasi-functoriality of $\widetilde{\mathcal{M}}$ in Example 2.12

It is easy to check that $\widetilde{\mathcal{M}}$ is a lax extension that preserves diagonals. So we only prove the quasi-functoriality. Take any two relations $R : X \nrightarrow Z$ and $S : Z \nrightarrow Y$. We need to show that for all $(\alpha, \beta) \in \widetilde{\mathcal{M}}$, if there are γ_R and γ_S in $\mathcal{M}Z$, with $(\alpha, \gamma_R) \in \widetilde{\mathcal{M}}R$ and $(\gamma_S, \beta) \in \widetilde{\mathcal{M}}S$, then there is a $\gamma \in \mathcal{M}Z$ such that $(\alpha, \gamma) \in \widetilde{\mathcal{M}}R$ and $(\gamma, \beta) \in \widetilde{\mathcal{M}}S$.
From the assumption that $(\alpha, \gamma_R) \in \widetilde{\mathcal{M}}R \subseteq \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}R$, we get that:

$$\forall A \in \alpha, \exists U_a \in \gamma_R \text{ s.t. } (A, U_a) \in \overleftarrow{\mathcal{P}}R.$$

Similarly we get the followings:

$(\alpha, \beta) \in \widetilde{\mathcal{M}}(R; S) \subseteq \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}(R; S)$ imply that:

$$\forall A \in \alpha, \exists V_A \in \beta \text{ s.t. } (A, V_A) \in \overleftarrow{\mathcal{P}}(R; S).$$

$(\gamma_S, \beta) \in \widetilde{\mathcal{M}}S \subseteq \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}S$ and $(\alpha, \beta) \in \widetilde{\mathcal{M}}(R; S) \subseteq \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}(R; S)$ implies that:

$$\forall B \in \beta, \exists U_B \in \gamma_S \text{ and } \exists A_B \in \alpha \text{ s.t. } (U_B, B) \in \overrightarrow{\mathcal{P}}S \text{ and } (A_B, B) \in \overrightarrow{\mathcal{P}}(R; S).$$

From $(A, V_A) \in \overrightarrow{\mathcal{P}}(R; S)$ it follows that:

$$\forall v \in V_A, \exists a_v \in A \text{ s.t. } (a_v, v) \in (R; S),$$

so there is a $z_v \in Z$ such that $(a_v, z_v) \in R$ and $(z_v, v) \in S$.
Now we define for every $A \in \alpha$:

$$U'_A = U_A \cup \{z_v \in Z \mid v \in V_A\}.$$

We claim that $(A, U'_A) \in \overleftarrow{\mathcal{P}}R$. To prove this, take $u \in U'_A$, we need to show that there exists $t \in A$ such that $(t, u) \in R$. Since $u \in U'_A$, we have two cases:

(i) $u \in U_A$, then from $(A, U_A) \in \overleftarrow{\mathcal{P}}R$ we are done.
(ii) $u \in \{z_v \in Z \mid v \in V_A\}$. In this case from the definition of z_v we have that there exists $a_v \in A$ such that $(a_v, z_v) \in R$.

On the other hand because for all $v \in V_A$, $(z_v, v) \in S$, we have that $(U'_A, V_A) \in \overleftarrow{\mathcal{P}}S$. We can similarly define for every $B \in \beta$ a set U'_B such that:

$$(U'_B, B) \in \overrightarrow{\mathcal{P}}S \text{ and } (A_B, U'_B) \in \overrightarrow{\mathcal{P}}R.$$

Now we are ready to introduce $\gamma \in \mathcal{M}Z$:

$$\gamma = \{U \subseteq Z \mid \exists A \in \alpha \text{ with } U'_A \subseteq U \text{ or } \exists B \in \beta \text{ with } U'_B \subseteq U\}.$$

It is clear that γ is upward closed, so $\gamma \in \mathcal{M}Z$. It is left to show that $(\alpha, \gamma) \in \widetilde{\mathcal{M}}R$ and $(\gamma, \beta) \in \widetilde{\mathcal{M}}S$. We have that $(\alpha, \gamma) \in \widetilde{\mathcal{M}}R$ iff $(\alpha, \gamma) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}R$ and $(\alpha, \gamma) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}R$. For the proof of $(\alpha, \gamma) \in \overrightarrow{\mathcal{P}}\overleftarrow{\mathcal{P}}R$ note that for every $A \in \alpha$ we have that $(A, U'_A) \in \overleftarrow{\mathcal{P}}R$. For $(\alpha, \gamma) \in \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}R$, pick $U \in \gamma$. Then from the definition of γ it follows that $U'_A \subseteq U$ for $A \in \alpha$ or $U'_B \subseteq U$ for $B \in \beta$. In the first case consider that $U_A \subseteq U'_A \subseteq U$ and by the assumption $(\alpha, \gamma_R) \in \widetilde{\mathcal{M}}R \subseteq \overleftarrow{\mathcal{P}}\overrightarrow{\mathcal{P}}R$ we get that there exists $T \in \alpha$ such that $(T, U_A) \in \overrightarrow{\mathcal{P}}R$, so $(T, U) \in \overrightarrow{\mathcal{P}}R$. For the case that there exists $B \in \beta$ such that $U'_B \subseteq U$, we have that $(A_B; U'_B) \in \overrightarrow{\mathcal{P}}R$.

Proof of Proposition 2.13

For the proof of (1) consider the following argument: Let $\pi_X : R \to X$ and $\pi_Z : R \to Z$ denote the projection maps. Since $R = (\pi_X)^\circ; \pi_Z$ and $LR = L((\pi_X)^\circ; \pi_Z)$, from quasi-functoriality of L it follows that

$$L(\pi_X)^\circ; L\pi_Z = L((\pi_X)^\circ; \pi_Z) \cap (DomL(\pi_X)^\circ \times Rng(L\pi_Z)).$$

But since $R = (\pi_X)^\circ; \pi_Z$ is full on both sides, the projection maps π_X and π_Z are surjective. It then follows that $\mathsf{T}\pi_X : \mathsf{T}R \to \mathsf{T}X$ and $\mathsf{T}\pi_Z : \mathsf{T}R \to \mathsf{T}Z$ are surjective, because set functors preserve surjective-ness. So $Rng(\mathsf{T}\pi_X) = Dom(\mathsf{T}\pi_X)^\circ = \mathsf{T}X$ and $Rng(\mathsf{T}\pi_Z) = \mathsf{T}Z$. Consequently we have

$$L((\pi_X)^\circ; \pi_Z) \cap \mathsf{T}X \times \mathsf{T}Z = L(\pi_X)^\circ; L\pi_Z,$$

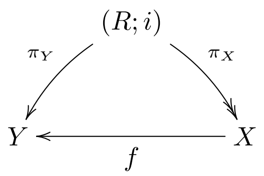
which simply implies $L((\pi_X)^\circ; \pi_Z) = L(\pi_X)^\circ; L\pi_Z$. Now in order to prove fullness of $LR = L((\pi_X)^\circ; \pi_Z)$ on $\mathsf{T}X$ and $\mathsf{T}Z$ it is sufficient to prove $L(\pi_X)^\circ; L\pi_Z : \mathsf{T}X \not\to \mathsf{T}Z$ is full on $\mathsf{T}X$ and $\mathsf{T}Z$. But we are done since

$$\begin{array}{rcll} \mathsf{T}X & = & Dom((\mathsf{T}\pi_X)^\circ; (\mathsf{T}\pi_Z)) & \mathsf{T}\pi_X \text{ is surjective} \\ & \subseteq & Dom(L(\pi_X)^\circ; (L\pi_Z)) & (L3) \end{array}$$

and

$$\begin{array}{rcll} \mathsf{T}Z & = & Rng((\mathsf{T}\pi_X)^\circ; (\mathsf{T}\pi_Z)) & \mathsf{T}\pi_Z \text{ is surjective} \\ & \subseteq & Rng(L(\pi_X)^\circ; (L\pi_Z)) & (L3) \end{array}$$

To prove (2) notice that $R; i \subseteq X \times Y$ is full on X, so by axiom of choice there exists a map $f : X \to Y$ such that $f \subseteq (R; i)$. Hence we get $\mathsf{T}f \subseteq Lf \subseteq L(R; i)$, and because $\mathsf{T}f$ is full on $\mathsf{T}X$, $L(R; i)$ is also full on $\mathsf{T}X$.



For the proof of (3) we refer to [14, Proposition 2] (where if fact it is stated that (3) holds for every lax extension L).

Proof of Fact 4.2

Here we will give a proof for item (v). Suppose that L_1 is a quasi-functorial lax extension for T_1 and L_2 is a functorial lax extension for T_2. We claim that L_1L_2 is a quasi-functorial lax extension for $\mathsf{T}_1 \circ \mathsf{T}_2$. First observe that since L_1 and L_2 are lax extensions, L_1L_2 is also a lax extension. Take $(\alpha, \beta) \in L_1L_2(R; S) \cap Dom(L_1L_2R) \times Rng(L_1L_2S)$. We get that $(\alpha, \beta) \in Dom(L_1(L_2R)) \times Rng(L_1(L_2S))$ and by functoriality of L_2, $(\alpha, \beta) \in L_1(L_2R; L_2S)$. Now from quasi-functoriality of L_1 we get that:

$$(\alpha, \beta) \in L_1(L_2R); L_1(L_2S) = L_1L_2R; L_1L_2S.$$

Proof of Proposition 4.2

1. Define $(\mathbb{A}', a'_I) := (A \cup \{a_\top\}, \Delta', \Omega', a'_I)$ such that $a'_I = a_I$ and for all $a \in A$, $\Delta(a) = \Delta'(a)$ and $\Omega(a) = \Omega'(a)$. For $(a_\top, c) \in A \times C$ define $\Delta'(a_\top, c) := \mathsf{T}(\{a_\top\})$ and $\Omega'(a_\top) := 0$. Since it is not difficult to check the equivalence of these automata, we leave it for the reader.
2. We will define $(\mathbb{A}', a'_I)$ over C by just removing the unsatisfiable elements of any $\Delta(a, c)$. $(\mathbb{A}', a'_I) = (A, \Delta', \Omega, a_I)$, where $\Delta'(a, c) = \{\phi \in \Delta(a, c) \mid \phi \text{ is a satisfiable element}\}$. $(\mathbb{A}', a_I)$ and $(\mathbb{A}, a_I)$ are equivalent since $\exists$ will never go through unsatisfiable elements in winning plays.
3. Take the coproduct of all witnessing coalgebra $\mathbb{Q}_\phi$ for all $\phi \in \Delta(a, c)$ and all $(a, c) \in A \times C$. The relation Y is the union of all Y_ϕ.